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FROM SIMPLE GROWTH TO NUMERICAL SIMULATIONS: A PRIMER IN DYNAMIC PROGRAMMING

Gianluca Femminis

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FROM SIMPLE GROWTH TO NUMERICAL SIMULATIONS: A PRIMER IN DYNAMIC PROGRAMMING

GIANLUCA FEMMINIS

ABSTRACT. These notes provide an intuitive introduction to dynamic programming. The first two Sections present the standard deterministic Ramsey model using the Lagrangian approach. These can be skipped by whom is already acquainted with this framework. Section 3 shows how to solve the well understood Ramsey model by means of a Bellman equation, while Section 4 shows how to "guess" the solution (when this is possible). Section 5 is devoted to applications of the envelope theorem. Section 6 provides a "paper and pencil" introduction to the numerical techniques used in dynamic programming, and can be skipped by the uninterested reader. Sections 7 to 9 are devoted to stochastic modelling, and to stochastic Bellman equations. Section 10 extends the discussion of numerical techniques. An Appendix provides details about the Matlab routines used to solve the examples.

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1. Utility maximization in a finite-horizon deterministic setting

One of the ingredients that we find in almost any growth model is the analysis of the agent's consumption behavior. In fact consumption, through savings, determines capital accumulation, which, in turn is one of the key "engines of growth". In this Section, we consider the problem of the optimal determination of consumption in the easiest possible framework, in which the lifetime of a single consumer is of a finite and known length. We solve this intertemporal problem using the Lagrangian approach: once the problem is well understood, it shall be easy to consider its infinite horizon counterpart and then to solve it by means of the dynamic programming approach. This shall be done in Sections 2 and 3, respectively.

1.1. **The problem.**

In our settings, a single consumer aims at maximizing her utility over her finite lifetime (hence, the consumer's horizon is finite). Time is "discrete", i.e., it is divided into periods of fixed length (say, a year or a quarter), and our consumer is allowed to decide her consumption level only once per period. The consumption goods she enjoys are produced by means of a "neoclassical" production function.[1]

We suppose that our consumer optimizes from time 0 onwards, and that her preferences are summarized by the following intertemporal utility function:

$$W_0 = \sum_{t=0}^{T} \beta^t U(c_t), \tag{1.1}$$

where $\beta \in (0, 1)$ is the subjective discount parameter, c_t is consumption at time t, and $T + 1$ is the length (in periods) of our consumer's lifetime. As for the single period utility function, $U(c_t)$, we accept the standard "neoclassical" assumptions, requiring that, in every period, the marginal utility is positive but decreasing, i.e. that $U'(c_t) > 0$, and $U''(c_t) < 0$. Moreover, we assume that: $\lim_{c_t \to 0} U'(c_t) = \infty$.

[1] An analysis concerning a *single* consumer may seem very limited. In particular, as it will become clear in a while, a single agent–being alone–optimizes under the constraint of the production function. This appears to be in sharp contrast with what happens in the real world. In fact, in a market economy, any optimizing consumer takes account of prices, wages, interest rates...
However, it can be shown that *if* markets are competitive and agents are all alike, the resources allocation in our exercises is equivalent to the allocation or resources that is achieved by a decentralized economy. Hence, while our *if* is a rather big one, our analysis is less limited than what it might seem at first sight.

Output (y_t) is obtained by means of a production function, the argument of which is capital (k_t): [2]

$$y_t = f(k_t). \tag{1.2}$$

As any well-behaved "neoclassical" production function, Eq. (1.2) satisfies some conditions, that are:

a: $f'(k_t) > 0$,
b: $f''(k_t) < 0$,

(in words, the marginal productivity of capital is positive, but decreasing),

c: $f(0) = 0$,

(this means that capital is essential in production).

d: $f'(0) > \delta + 1/\beta - 1$,
e: $\lim_{k_t \to \infty} f'(k_t) = 0$,

(as it will become clear in what follows, hypotesis (d) implies that capital – at least at its lowest level – is productive enough to provide the incentive for building a capital stock, while assumptions (e) rules out the possibility that capital accumulation goes on forever.) [3]

At this point, it is commonly assumed that output can be either consumed or invested, i.e. that $y_t = c_t + i_t$ (which implies that we are assuming away government expenditure). When capital depreciates at a constant rate, δ, the stock of capital owned by our agent in period 1 is: $k_1 = i_0 + (1-\delta)k_0$. Accordingly, Eq. (1.2) and the output identity, $y_t = c_t + i_t$, imply that k_1 can be written as:

$$k_1 = f(k_0) + (1-\delta)k_0 - c_0.$$

Hence, in general, we have that

$$k_{t+1} = f(k_t) + (1-\delta)k_t - c_t, \tag{1.3}$$

[2] If you feel disturbed by the fact that capital is the unique productive input, consider that we can easily encompass a fixed supply of labour in our framework. We might have specified our production function as $y_t = g(k_t, \bar{l})$, where $\bar{l}$ is the labour fixed supply; in this case we could have normalized $\bar{l}$ to unity and then we could have written $f(k_t) \equiv g(k_t, 1)$. An alternative, and more sophisticated, way of justifying Eq. (1.2) is to assume that output is obtained by means of a production function which is homogeneous of degree one, so that there are constant returns to scale. Then, one interprets k_t as the capital/labour ratio.

[3] What is really necessary is to accept that $\lim_{k_t \to \infty} f'(k_t) < \delta$, an hypothesis that can hardly be considered restrictive. The assumption in the main text allows for a slightly easier exposition.

for $t = 0, 1, \ldots, T$. In addition to the above set of dynamic constraints, we require that

$$k_{T+1} \geq 0. \tag{1.4}$$

In words, this obliges our consumer to end her life with a non-negative stock of wealth. This condition must obviously be fulfilled by a consumer which lives "in insulation" (a negative level of capital stock does not make any sense in this case); if our agent is settled in an economic system where financial markets are operative, what we rule out is the possibility that our consumer dies in debt.

Summing up, we wish to solve the problem:

$$\max W_0 = \max \sum_{t=0}^{T} \beta^t U(c_t),$$

under the T constraints of the (1.3)-type and under constraint (1.4). Notice that the solution of the problem requires the determination of $T+1$ consumption levels $(c_0, c_1, \ldots., c_T)$, and of $T+1$ values for the capital stock $(k_1, k_2, \ldots., k_{T+1})$.

We can approach the consumer's intertemporal problem by forming a "present value" Lagrangian:

$$\begin{aligned} L_0 \quad = \quad & U(c_0) + \beta U(c_1) + \beta^2 U(c_2) + \ldots + \beta^T U(c_T) \\ & -\lambda_0[k_1 - f(k_0) - (1-\delta)k_0 + c_0] \\ & -\beta\lambda_1[k_2 - f(k_1) - (1-\delta)k_1 + c_1] \\ & -\ldots \\ & -\beta^{T-1}\lambda_{T-1}[k_T - f(k_{T-1}) - (1-\delta)k_{T-1} + c_{T-1}] \\ & -\beta^T\lambda_T[k_{T+1} - f(k_T) - (1-\delta)k_T + c_T] \\ & +\beta^T \mu k_{T+1}. \end{aligned} \tag{1.5}$$

To solve the problem, we must differentiate (1.5) with respect to c_t, k_{t+1}, λ_t (for $t = 0, 1, \ldots, T$), and with respect to μ.[4]

The first order conditions with respect to the $T+1$ consumption levels are:

[4]The condition we imposed on the single period utility function and on the production function guarantee that we obtain a global maximum. See, e.g. Beavis and Dobbs [1990], or de la Fluente [2000].

$$
\begin{aligned}
U'(c_0) &= \lambda_0 \\
U'(c_1) &= \lambda_1 \\
... &= ... \\
U'(c_t) &= \lambda_t \\
... &= ... \\
U'(c_T) &= \lambda_T.
\end{aligned} \tag{1.6}
$$

Notice that each Lagrange multiplier λ_t expresses the consumer's marginal utility of consumption, as perceived in the future period t.

When our agent optimizes with respect to the $T+1$ capital levels (from k_1 to k_{T+1}), she obtains:

$$
\begin{aligned}
\lambda_0 &= \beta\lambda_1[f'(k_1) + (1-\delta)] \\
\beta\lambda_1 &= \beta^2\lambda_2[f'(k_2) + (1-\delta)] \\
&...... \\
\beta^t\lambda_t &= \beta^{t+1}\lambda_{t+1}[f'(k_{t+1}) + (1-\delta)] \\
&...... \\
\beta^{T-1}\lambda_{T-1} &= \beta^T\lambda_T[f'(k_T) + (1-\delta)] \\
\beta^T\mu &= \beta^T\lambda_T.
\end{aligned} \tag{1.7}
$$

Of course, derivation of (1.5) with respect to the Lagrange multipliers $\lambda_t,\ t = 0, 1, 2, ..., T$ yields the set of constraints (1.3). Finally, derivation of (1.5) with respect to μ gives the constraint (1.4); in addition one must consider the "complementary slackness" condition:

$$\beta^T\mu k_{T+1} = 0 \text{ and } \mu \geq 0, \tag{1.8}$$

which shall be commented upon in a while.

1.2. **The Euler equation.** Consider now any first order condition belonging to group (1.7): one immediately sees that those equations can be manipulated using the appropriate first order conditions of group (1.6). The typical result of a practice of this kind is:

$$U'(c_t) = \beta U'(c_{t+1})[f'(k_{t+1}) + (1-\delta)] \tag{1.9}$$

This condition is known as the Euler equation, which is of remarkable importance not only to understand many growth models but also in consumption theory.

The Euler equation (1.9) tells us that an optimal consumption path must be such that – in any period – the marginal utility for consumption is equal to the following period marginal utility, discounted by β and capitalized by means of the net marginal productivity of capital. To gain some intuition about the economic meaning of Eq. (1.9), consider that it can be interpreted as prescribing the equality between the marginal rate of substitution between period t and period $t+1$ consumptions (i.e. $U'(c_t)/\beta U'(c_{t+1})$), and the marginal rate of transformation, $f'(k_{t+1}) + (1-\delta)$.[5]

To improve your understanding of this point, pick a consumption level for period t, say $\tilde{c}_t$, then choose a consumption level for the subsequent period $t+1$, say $\tilde{c}_{t+1}$.[6] Suppose that the latter level, $\tilde{c}_{t+1}$, does not satisfy the Euler equation: we require only that it is feasible, i.e. that it can be produced given the capital stock implied by $\tilde{c}_t$. In deciding whether to consume $\tilde{c}_t$ in period t and $\tilde{c}_{t+1}$ in period $t+1$, our consumer must consider what would happen to her overall utility if she decided to increase the time t consumption by a small amount ξ. In this case, her time t utility would increase by (approximately) $U'(\tilde{c}_t)\xi$.[7] Moreover, because her savings would decrease by ξ, her next period resources would decrease by $[f'(k_{t+1}) + (1-\delta)]\xi$, that is, by ξ multiplied by the productivity of the "marginal" savings. The reduction in period $t+1$ utility is given by $U'(\tilde{c}_{t+1})[f'(k_{t+1}) + (1-\delta)]\xi$. From the perspective of time t, this variation in utility must be discounted; hence, its period t value is $\beta U'(\tilde{c}_{t+1})[f'(k_{t+1}) + (1-\delta)]\xi$.

If $U'(\tilde{c}_t)\xi > \beta U'(\tilde{c}_{t+1})[f'(k_{t+1}) + (1-\delta)]\xi$, it is convenient to increase period t consumption: the utility gain in that period is larger than the utility loss suffered at time $t+1$ once this is discounted back to period t.

Likewise, if $U'(\tilde{c}_t)\xi < \beta U'(\tilde{c}_{t+1})[f'(k_{t+1}) + (1-\delta)]\xi$, it is convenient to decrease period t consumption: the utility loss in that period is smaller than

[5] An alternative interpretation is based on the fact that our representative consumer – forsaking one unit of consumption today – obtains $f'(k_{t+1}) + (1-\delta)$ unit of consumption tomorrow. Accordingly, $f'(k_{t+1}) + (1-\delta)$ can also be interpreted as the price of current consumption if the price of *future* consumption is conceived as the numeraire, and hence fixed to unity. According to this interpretation, Eq. (1.9) can be seen as prescribing the equalization of the marginal rate of substitution $U'(c_t)/\beta U'(c_{t+1})$ with the price ratio $[f'(k_{t+1}) + (1-\delta)]/1$.

[6] In these notes, we denote by a twiddle an arbitrary level for a variable, with a star an optimal level, and by a hat a steady state level for that variable. A steady state is a point such that every dynamic variable does not change over time.

[7] If you do not "see" this, consider that the difference (in terms of period t utility) of the two policies is $U(\tilde{c}_t + \xi) - U(\tilde{c}_t)$. Applying Taylor's theorem to $U(\tilde{c}_t + \xi)$ one obtains: $U(\tilde{c}_t + \xi) \simeq U(\tilde{c}_t) + U'(\tilde{c}_t)\xi$.

the discounted utility gain. In this case, one has better to reduce period t consumption.

From this reasoning, we can convince ourselves that Eq. (1.9) must hold true when the consumption sequence is optimally chosen.

The Euler equation is useful to relate the evolution of consumption over time to the existing capital stock.

Assume that $c_{t+1} = c_t$, so that $\Delta c_{t+1} = 0$,[8] and notice that $c_{t+1} = c_t$ implies $U'(c_t) = U'(c_{t+1})$. From Eq. (1.9), a constant consumption can be optimal if and only if $k_{t+1} = \hat{k}$, where $\hat{k}$ is such that:

$$1 = \beta[f'(\hat{k}) + (1 - \delta)]. \tag{1.10}$$

In the steady state, i.e. when $k_t = \hat{k}$, and consumption is constant over time, the impatience parameter β exactly offsets the positive effects on savings exerted by the fact that they are rewarded by the marginal productivity of capital. [9]

Whenever $k_{t+1} < \hat{k}$, the marginal productivity of capital is higher than at $\hat{k}$ (i.e. $f'(k_{t+1}) > f'(\hat{k})$);[10] hence $\beta[f'(k_{t+1}) + (1 - \delta)] > 1$. Therefore, the Euler equation is satisfied only for consumption levels such that $U'(c_t) > U'(c_{t+1})$. Hence, it must be true that $c_t < c_{t+1}$. In words, since the marginal productivity of capital is high, saving is very rewarding. Therefore, it is sensible to save a lot, by reducing consumption at the "early" date t. Because the "early" consumption is low, consumption increases over time.

By the same token, when $k_{t+1} > \hat{k}$, capital is "abundant" and its marginal productivity gets low. Therefore $\beta[f'(k_{t+1}) + (1 - \delta)] < 1$. Eq. (1.9) is satisfied if $U'(c_t) < U'(c_{t+1})$, which implies that $c_t > c_{t+1}$. Because the marginal productivity of capital is low, saving is ill-compensated. Therefore, it is sensible to choose a high consumption level at t, and decrease it over time.

Finally, notice that Eq. (1.9)$-$relating the period t consumption level to the one in period $t+1-$applies for $t = 0, 1, ..., T-1$ (consumption at time $T+1$ does not make sense by assumption). Hence, the Euler equation provides us with T relations that we exploit when we wish to solve analytically our maximization problem.

1.3. **The solution.** What we wish to determine are the $T+1$ consumption values (i.e. $c_0, c_1, ..., c_T$) and the $T+1$ capital stocks (i.e. $k_1,\ k_2,\ ...,\ k_{T+1}$).

---

[8]Following the convention often used in time series analysis, we denote, for any variable y_t, $\Delta y_t \equiv y_t - y_{t-1}$.

[9]Notice that the uniqueness of $\hat{k}$ is granted by assumptions (a), (b), and (d).

[10]This is granted by the assumption of positive but *decreasing* marginal productivity of capital, (a) and (b).

As already remarked, the Euler equation (1.9) provides us with T relations; the constraints like (1.3) are $T+1$. Accordingly, to close the model we need a further equation.

This is obtained starting from the complementary slackness condition (1.8). This tells us that, if the consumer uses up her entire capital stock in the final period, so that $k_{T+1} = 0$, then μ is different from 0; alternatively, if $\mu = 0$, the final capital stock is positive. However, the last condition in (1.7) tells us that μ must be positive, since it is equal to λ_T, and λ_T is the marginal utility of consumption at time T, which can not be nought. Hence, the final period capital must be zero, which is very sensible in economic terms: because our agent will not consume anything in period $T+1$, it is pointless for her to keep some capital: she can always improve her overall utility by eating up this stock of resources. In sum, we can be sure that $k_{T+1} = 0$.

This is the equation that closes our model. The system composed by T equation like (1.9), of $T+1$ equations like (1.3), and by $k_{T+1} = 0$ can – at least in principle – be solved for the $T+1$ consumption levels and for the $T+1$ capital levels.

Bear in mind that the complementary slackness condition (1.8) can be reformulated, by means of the last conditions in (1.7) and (1.6), as:

$$\beta^T U'(c_T) k_{T+1} = 0. \tag{1.11}$$

2. The infinite-horizon consumption-growth problem

Let us now imagine that our agent is going to live forever. At first sight, this might seem crazy. However, we may conceive our agent as a person who cares about her offsprings. In this case, she should consider that her sons and daughters will be concerned about their offsprings' welfare and so on. Hence, she should optimize over the entire future horizon. If you wish, you can think about our agent not as a person, but as a dynasty. An alternative interpretation of the model we are about to present, is that the optimizing agent actually is a social planner who aims at maximizing a social welfare function whose arguments are the discounted utilities of the agents who are alive now and in any possible future date.

Assuming that our agent optimizes from time 0 onwards, her preferences are now given by the following intertemporal utility function:

$$W_0 = \sum_{t=0}^{\infty} \beta^t U(c_t), \tag{2.1}$$

The above expression is analogous to (1.1), but for the fact that now the agent's horizon extends up to infinity.

Our problem is now to

$$\max W_0 = \max \sum_{t=0}^{\infty} \beta^t U(c_t);$$

but, while the constraints are given by equations such as (1.3), the fact that the consumer's planning horizon is infinite implies that we cannot impose a "terminal constraint" like (1.4).

In this case, the consumer's problem is tackled by means of the following "present value" Lagrangian:

$$\begin{aligned} L_0 &= U(c_0) + \beta U(c_1) + ... + \beta^t U(c_t) + ... \\ &\quad -\lambda_0[k_1 - f(k_0) - (1-\delta)k_0 + c_0] \\ &\quad -... \\ &\quad -\beta^t \lambda_t[k_{t+1} - f(k_t) - (1-\delta)k_t + c_t] \\ &\quad -.... \end{aligned} \tag{2.2}$$

Problem (2.2) differs from (1.5) because it involves an infinite number of discounted utility terms, and an infinite number of dynamic constraints; moreover – as already remarked – the constraint concerning the final level for the stock variable is missing.

We optimize (2.2) with respect to c_t, k_{t+1}, and λ_t for $t = 0, 1, ...$ obtaining:

$$U'(c_t) = \lambda_t, \ \forall \ t, \tag{2.3}$$

$$\beta^t \lambda_t = \beta^{t+1} \lambda_{t+1}[f'(k_{t+1}) + (1-\delta)], \ \forall \ t, \tag{2.4}$$

and, of course:

$$k_{t+1} = f(k_t) + (1-\delta)k_t - c_t, \ \forall \ t. \tag{2.5}$$

These conditions are necessary, but they are not sufficient: a "final condition" is missing. In our infinite horizon model, the role of this final condition is played by the so-called transversality condition (henceforth tvc), which–in this set up–is:

$$\lim_{T\to\infty} \beta^T U'(c_T) k_{T+1} = 0. \tag{2.6}$$

Comparing the above expression with (1.11), we immediately notice that (2.6) is the limit, for $T \to \infty$, of (1.11). This suggests us that the tvc plays the role of the "missing" terminal condition. The tvc has a clear economic interpretation: it rules out policies implying a "too fast" capital accumulation in the long run. To understand this point, assume that our consumer follows a policy implying a growing capital stock. Of course, this capital would be accumulated at the expenses of consumption. Hence, this policy would imply a high marginal utility of consumption, and an high and increasing value for the product $U'(c_T)k_{T+1}$. This, in itself, does not involve the violation of condition (2.6). In fact, condition (2.6) allows $U'(c_T)k_{T+1}$ to increase over time. Nevertheless, this growth must be slow enough to be compensated by the convergence to 0 of the term β^T. This is why we say that the transversality condition rules out policies implying a "too fast" long-run capital accumulation.

2.1. **The qualitative dynamics.** In our model, as it happens in many infinite horizon frameworks, it is useful to draw the phase diagram. To do this, we first consider the stability loci for each of the two variables (i.e. we compute where $\Delta c_{t+1} = 0$ and $\Delta k_{t+1} = 0$). This will help us to understand how consumption and capital change over time whenever they are not on their stability loci. Finally, we will jointly consider our knowledge for the dynamics of the two variables.

As for consumption, combining equation (2.4) with (2.3), we immediately see that optimality requires that the Euler equation (1.9) is satisfied. We already know that the Euler equation is useful to describe the evolution of consumption over time. In particular, we know that when $k_t = \hat{k}$, then $c_{t+1} = c_t$ and hence $\Delta c_{t+1} = 0$ (consider again how we obtained Eq. (1.10)). Hence, in Figure 1, we plot the locus implying stationarity for consumption as a the vertical line drawn at $\hat{k}$.

[Insert Figure 1]

Whenever $k_t < \hat{k}$, it is optimal for our consumer to increase her consumption over time (hence, $c_t < c_{t+1}$). When $k_t > \hat{k}$, consumption must be shrinking over time ($c_{t+1} < c_t$). This behavior is summarized by the arrows in Figure 1.

From Eq. (2.5), we see that $\Delta k_{t+1} = f(k_t) - \delta k_t - c_t$, hence, capital is stationary when

$$c_t = f(k_t) - \delta k_t. \tag{2.7}$$

This relation can be portrayed as a function starting at the origin (by assumption (c)), with a maximum at $\underline{k}$ (defined as the capital level such that $f'(\underline{k}) = \delta$), and intersecting again the $c_t = 0$ axis at $\bar{k}$ (which is the capital level such that $c_t = 0$, i.e. it is obtained solving the equation $f(\bar{k}) - \delta\bar{k} = 0$).[11] The behavior of the $\Delta k_{t+1} = 0$ locus is portrayed in Figure 2.

[Insert Figure 2]

To see what happens when the economic system is not on the stability locus (2.7), pick a capital level $\tilde{k} \in [0, \bar{k}]$; the corresponding consumption level guaranteeing stationarity for capital obviously is

$$\tilde{c} = f(\tilde{k}) - \delta\tilde{k}.$$

If the consumer chooses a consumption level $c_t > \tilde{c}$, her capital stock must decrease over time: the consumption is so high that our consumer lives using up part of her capital. More precisely, consumption is higher than the level, $\tilde{c}$, guaranteeing that the difference between gross production, $f(\tilde{k})$, and consumption is exactly equal to capital depreciation $\delta\tilde{k}$. Therefore the capital stock must decrease. The converse happens when our consumer chooses a consumption level c_t that is lower than $\tilde{c}$: her capital increases over time because a consumption lower than $\tilde{c}$ implies that there is room for some savings, and hence there is some net investment. This behavior is summarized by the arrows in Figure 2.

Merging Figures 1 and 2, we obtain Figure 3, which summarizes the dynamics of the model.

[Insert Figure 3]

Notice that there are three steady states (E, the origin, and I). It is easy to see that E is a steady state: here the two loci $\Delta k_{t+1} = 0$ and $\Delta c_{t+1} = 0$ intersect. Notice that the consumption and capital level characterizing this steady state can be obtained solving the system

$$\left\{ \begin{array}{c} 1 = \beta[f'(\hat{k}) + (1 - \delta)] \\ \hat{c} = f(\hat{k}) - \delta\hat{k} \end{array} \right. . \tag{2.8}$$

From Eqs. (1.9) and (2.7) it is clear that at E both consumption and capital are not pressed to change over time.

The origin is a resting point because of Assumption (c): if capital is 0, there is no production and hence no possibility of further capital accumulation. This resting point is usually considered uninteresting.

[11]Existence and uniqueness of $\bar{k}$ are granted by assumptions (b) and (e).

It is less obvious that also I is a steady state. To gain some intuition about the reason why I is a steady state, consider that the marginal utility of consumption increases very rapidly as consumption approaches 0 (This is because we assumed that $\lim_{c_t \to 0} U'(c_t) = \infty$.) Hence, the increase in the marginal utility of consumption prescribed by Eq. (1.9) for $k_t > \hat{k}$ implies smaller and smaller reductions in consumption as c_t approaches 0. Hence consumption does not become negative (which would of course have no economic meaning) and I is a resting point.

A less heuristic argument is presented in the next three paragraphs, that can be skipped by the uninterested reader.

To see that I is a steady state, rewrite (1.9) as:

$$U'(c_{t+1}) = U'(c_t) \left[\frac{1}{\beta[f'(k_{t+1}) + (1-\delta)]} \right].$$

Pick a capital level, say $\tilde{k}$, belonging to the interval $(\hat{k}, \infty)$: it is easy to see that $f'(\tilde{k}) + (1-\delta) \in (1-\delta, 1/\beta)$: this comes from (1.10) and from the assumptions: $\lim_{k_t \to \infty} f'(k_t) = 0$, and $f''(k_t) < 0$. Therefore, when $\tilde{k} \in (\hat{k}, \infty)$, the term in the big square brackets in the equation above must be larger than one: the largest value for the denominator is "slightly smaller" than one. Therefore, the Euler equation not only tells us that the marginal utility of consumption must increase (and hence that consumption must decrease), but also that the rate of change of marginal utility is limited, being:

$$\frac{U'(c_{t+1}) - U'(c_t)}{U'(c_t)} = \frac{1}{\beta[f'(\tilde{k}) + (1-\delta)]} - 1.$$

The value of the right hand side of the expression above belongs to the interval $\left(0, \frac{1}{\beta(1-\delta)} - 1\right)$. This has a relevant implication: because the rate of change of the marginal utility is bounded, when we consider a sequence of consumption levels that – starting from a non negative value – fulfills the Euler equation, we see that this sequence cannot go to zero in finite time. This is because the marginal utility of consumption cannot "reach infinity" in finite time (bear in mind that $\lim_{c_t \to 0} U'(c_t) = \infty$). Because consumption takes an "infinite time" to reach its limiting value (that is, 0), in the meantime capital must reach $\bar{k}$ (since consumption decreases, the system must reach at some time the area below the $\Delta k_{t+1} = 0$ locus where capital grows, approaching $\bar{k}$). Hence I is a stationary state for our system.

Let us now consider that our consumer is constrained by the fact that her initial stock of capital is given (at k_0). Hence, in choosing her optimal consumption path, she must take into account this constraint. In Figure 3, we have depicted some of the possible paths that our agent may decide to follow. These paths are intended to fulfill the Euler equation (1.9) and the capital accumulation constraint (1.3).[12]

The fact that there are many (actually, infinitely many) trajectories that are compatible with one initial condition cannot be surprising: while the capital stock k_0 is given, our consumer is free to pick her initial consumption level, which then determines the path for consumption and capital (via equations (1.9) and (1.3)).

2.2. **The optimal path.** So far, we have seen that there are multiple paths compatible with the same initial condition. What we need to do now, is to select the optimal one(s).

First, notice that the trajectory starting at A in Figure 3 cannot be optimal: because consumption is ever-increasing, capital must go to zero in finite time. Since by assumption capital is essential in production, at that time consumption must collapse, becoming nought. This big jump in consumption violates the Euler equation (which, to be fulfilled, would require a further increase in consumption). In fact, the path starting at A – ad all the paths akin to this one – cannot be optimal.

Second, consider the trajectory starting at B. In this case, our consumer chooses *exactly* the consumption level that leads the system to the stationary point E. This path not only fulfills the difference equations (1.9) and (1.3) but also the transversality condition (2.6). In fact, as time goes to infinity (i.e. as $t \to \infty$), capital and consumption approach their steady state levels, $\hat{k}$ and $\hat{c}$, which are given by System (2.8). The fact that the long run levels for consumption and capital are positive and constant, tells us that, in the steady state, the marginal utility of consumption is finite. Hence, $\lim_{t\to\infty} \beta^t U'(c_t) k_{t+1} = \lim_{t\to\infty} \beta^t U'(\hat{c})\hat{k} = 0$ simply because $\beta < 1$, and this second trajectory is optimal.

Third, consider the trajectory starting at C and leading to I. In this case, consumption and capital first increase together, but then consumption (as capital becomes larger than $\hat{k}$) starts to shrink while capital is still accumulated. In the long run, our optimizing agent finds herself around I, devoting all the productive effort to maintain an excessive stock of capital, which is actually never used to produce consumption goods. Clearly following this trajectory cannot be optimal, and the transversality condition is violated, because the marginal utility of consumption tends to be infinite.

[12] It is not possible to check that our paths in Figure 3 conform exactly to what is prescribed by our difference equation. However, notice that they have been drawn respecting the "arrows" that have been obtained from the difference equations (1.9) and (1.3).

In the next two paragraphs, that can be skipped, we give a less heuristic idea of the reasons why a path like the one starting at C violates the tvc.

To check whether a path of this type fulfills the tvc, imagine to be "very close" to I. Here, capital is (almost) $\bar{k}$, hence consumption should (approximately) evolve according to:

$$U'(c_{t+1}) = U'(c_t)\frac{1}{\beta[f'(\bar{k}) + (1-\delta)]}.$$

Therefore:

$$\beta U'(c_{t+1}) = U'(c_t)\frac{1}{[f'(\bar{k}) + (1-\delta)]} > U'(c_t),$$

where the latter inequality comes from the fact that for $k_t \in (\underline{k}, \bar{k})$, $f'(k_t) + (1-\delta) < 1$ (at $\underline{k}$, $f'(\underline{k}) = \delta$, and $f''(k_t) < 0$)). Because "around" $\bar{k}$, $\beta U'(c_{t+1}) > U'(c_t)$, $\lim_{t\to\infty} \beta^t U'(c_t) > 0$ (Suppose that at a given time T our system is already "very close" to I. In this case, in the following periods, i.e. for $t > T$, $\beta^{t-T}U'(c_t) > U'(c_T)$. Hence $\lim_{t\to\infty} \beta^t U'(c_t) = \lim_{t\to\infty} \beta^T \beta^{t-T} U'(c_t) > \beta^T U'(c_T)$) Hence, the tvc (2.6) is not fulfilled and any trajectory like the one starting at C is not optimal.

Summing up, the *unique* optimal path is the one leading to the steady state E; this path prescribes a monotonic increasing relation between consumption and capital. Our consumer (or our economic system) "jumps" on this path by adjusting the initial consumption to the level compatible with the existing capital stock, and with the behavior prescribed by our optimal trajectory.

3. The Dynamic Programming formulation

In this Section, we solve the infinite-horizon growth model exploiting the dynamic programming approach: we shall take advantage of our previous understanding of the solution to introduce this new technique in an intuitive way.

Consider again the intertemporal utility function (2.1). Obviously, our optimizing agent's preferences can be written as:

$$W_0 = U(c_0) + \sum_{t=1}^{\infty} \beta^t U(c_t).$$

In the formulation above, we have "separated" the utility obtained in the current period, 0, from the ones that will be grasped in the future, but there is no change in the meaning for W_0.

Notice that the expression above can be reformulated as:

$$W_0 = U(c_0) + \beta \left[\sum_{t=1}^{\infty} \beta^{t-1} U(c_t) \right]. \tag{3.1}$$

Here, we have collected the factor β that is common to all the addenda expressing future utilities. The reason for this manipulation is that the term in the big square brackets represents the consumer's preferences from the perspective of time 1.

The problem we want to solve now is the very same we faced in the previous Section: we wish to

$$\max W_0 = \max \sum_{t=0}^{\infty} \beta^t U(c_t);$$

under the constraints given by equation (1.3), for $t = 0, 1, ...$. Hence, we wish to determine the optimal consumption c_t^*, and k_{t+1}, for $t = 0, 1, ...$

Equation (3.1) allows us to write our problem as:

$$\max W_0 = \max \left\{ U(c_0) + \beta \left[\sum_{t=1}^{\infty} \beta^{t-1} U(c_t) \right] \right\}. \tag{3.2}$$

Now consider, in Figure 4, the optimal path starting from B and approaching E. This trajectory represents a function, say $\varphi(.)$, relating optimal consumption to the same period capital stock. We mean that c_1^* can be expressed as $c_1^* = \varphi(k_1)$; that c_2^* can be viewed as $c_2^* = \varphi(k_2)$, and so on. The function $c_t^* = \varphi(k_t)$ is unknown, and it can be very complex; actually what usually happens is that our function $\varphi(.)$ cannot be expressed in a closed−form analytical way. Nevertheless, the point that we underscore here is that Figure 4 powerfully supports the idea that we have just stated, i.e. that we can consider the optimal consumption as a function of contemporaneous capital. It is worth underscoring that our function $c_t^* = \varphi(k_t)$ is "stationary": it is always the same function, irrespective of the time period we are considering. Hence, the time dimension of the problem disappears. The intuition to understand why $\varphi(.)$ is independent of time is to consider our consumer's perception of the future: because she lives forever, at time 0 her horizon is infinite, and so it is at time 1. Hence, she must not ground her decision on time, but just on capital, which therefore is the unique state variable in our model.

[Insert Figure 4]

Notice that we have not proved that $\varphi(.)$ is continuous and differentiable. However, the evolution of capital and consumption on the optimal path must fulfill equations (1.9) and (1.3), which are continuos and differentiable. Hence, we have "good reasons to believe" that $\varphi(.)$ actually is continuous and differentiable, and we skip the formal proof for these statements.

Now consider that if we can write $c_1^* = \varphi(k_1)$, then we are also able to express the capital stock at time 2 as a function of k_1: in fact, from (1.3),

$$k_2 = f(k_1) + (1-\delta)k_1 - c_1^*,$$

hence, on the optimal path:

$$k_2 = f(k_1) + (1-\delta)k_1 - \varphi(k_1) = \phi(k_1).$$

The fact that $k_2 = \phi(k_1)$, allows us to consider the time 2 consumption as a function of k_1: $c_2^* = \varphi(k_2) = \varphi(\phi(k_1))$.

The important point here is to realize that *we can iterate this reasoning to express the whole sequence of optimal consumptions as a function of the time* 1 *capital stock.* (In fact, in general, $k_{t+1} = \phi(k_t)$, and $c_{t+1}^* = \varphi(k_{t+1}) = \varphi(\phi(k_t))$, and we can iterate the substitutions until we reach k_1).

Hence, when consumption is optimally chosen from period 1 onward, the group of addenda in the big square brackets in (3.2) can be expressed as a function of k_1 alone:

$$\sum_{t=1}^{\infty} \beta^{t-1} U(c_t^*) = V(k_1).$$

$V(k_1)$ is a "maximum value function": it represent the maximum lifetime utility that can be obtained in period 1, when all the consumption levels are optimally chosen given the available capital stock and the need to fulfil the constraints of the (1.3)-type.

Because the value of all the future choices can be summarized in the function $V(k_1)$, we can think about the consumer's intertemporal maximization problem in a way that is different from the initial one. We can imagine that, at time 0, she picks her optimal period 0 consumption, taking account of the fact that, given the available capital, an increase in current consumption reduces the future capital stock (via equation (1.3)), and therefore negatively affects the future overall utility $V(k_1)$.

Accordingly, the consumer's intertemporal problem can be written as:

$$\max_{c_0} \{U(c_0) + \beta V(k_1)\},$$
$$s.t.\ k_1 = f(k_0) + (1-\delta)k_0 - c_0,$$
$$k_0 \text{ given.}$$

Now, imagine that our consumer solves her period 0 constrained optimization problem, which means that she determines c_0^* as a function of k_0. Our representative consumer, obtaining c_0^*, determines also her time 0 maximum value function, $V(k_0)$, which means that

$$\begin{aligned} V(k_0) &= \max_{c_0} \{U(c_0) + \beta V(k_1)\}, \\ s.t.\ k_1 &= f(k_0) + (1-\delta)k_0 - c_0, \\ &k_0 \text{ given.} \end{aligned} \tag{3.3}$$

The above problem is said to be expressed as a "recursive procedure" or as a "Bellman equation".

While it is easy to understand that problem (3.3) gets its name from Bellman's [1957] book, it is not so simple to explain in plain English what a recursive procedure is.

Let us try. A procedure is recursive when one of the steps that makes up the procedure requires a new running of the procedure. Hence, a recursive procedure involves some degree of "circularity". As a simple example, consider the following "recursive" definition for the factorial of an integer number:

«if $n > 1$, then the factorial for n is $n! = n(n-1)!$;
when $n = 1$, then $1! = 1$».

Clearly, the above procedure defines $n!$ by means of $(n-1)!$, that is defined exploiting $(n-2)!$ and so on. This procedure goes on until 1 in reached, at this point the "termination clause", $1! = 1$, enters into stage.

The logic of problem (3.3) is quite similar: the maximum value $V(k_0)$ is obtained choosing the current consumption to maximize the sum of the current utility and of the next-period discounted maximum value, which, in turn is obtained by choosing future consumption in order to maximize the sum of the future period utility and of the two periods ahead maximum value ....

The difference between (3.3) and the factorial number example lies in the fact that our Bellman equation does not have a termination clause. This is due to the fact that the planning horizon for our representative agent is infinite. If she were to optimize only between period 0 and a given period T – for example, because she is bound to die at T – it would have been natural to introduce a terminal condition: in this case, the maximum value at time $T+1$, for any stock of capital k_{T+1}, would have been equal to nought (as discussed in Sub-section 1.3).

Notice that the example we have just sketched implies that the maximum value function depends not only on capital, but also on time. In fact, when the agent's time horizon is finite, the maximum value function typically depends on the remaining optimization horizon: how long you are going to be still alive typically matters a lot for you, and it also affects how you evaluate your stock of wealth. Accordingly, at time t the maximum value function is characterized by some terms involving $T-t$.

In contrast – as already underscored – in problem (3.3) we have written the maximum value function as depending only on capital. This sometimes strikes sensitive students: in fact, considering $V(k_t)$ as a *function of capital alone*, we imply that $V(k_t)$ does not change over time despite the fact that – moving backward from time 1 to time 0 – we discount the previous maximum value function, and then we add to it the term $U(c_0^*)$. In other words, in problem (3.3), we have the very same function $V(k_t)$ both on the left and on the right hand side.

The intuition to understand why $V(k_t)$ is independent of time is to consider again that *our consumer's perception of the future is the same at time* 0 *and at time* 1, simply because she lives forever. Hence, she bases her consumption decision only on capital, which therefore is the unique state variable in our model. Because the function $V(k_t)$ summarizes the optimal consumption decisions from period t onward, this function depends only on capital.

The independence of time of the maximum value function in infinite horizon frameworks is one of the reasons why these frameworks are so popular: their maximum value function – having a unique state variable – is less *complex* to compute. Notice however that the absence of a termination clause makes the problem of finding a solution conceptually more *difficult*. In fact, when we have a terminal time, we know the maximum value function for that period: this is just the final period utility function. Hence, we can always solve the problem for the terminal time, and then work "backward" toward the present. This simple approach is precluded in infinite horizon models. Solving a finite horizon problem is similar to decide how to send to the surface of the Moon a scientific pod, while the solution of an infinite horizon model is analogous to finding an optimal lunar orbit for the pod. In the first case, you can analyze all the possible location to pick the most convenient one, say the Tranquility sea. Then you realize that to send your pod to the Tranquility sea, you need a lunar module; then−working backward−you determine that a spaceship is needed to get close to the moon and finally you understand that it takes a Saturn−V missile to move the spaceship out of the Earth atmosphere. Notice that, in this case, you have a clear hint on how to start to work out your sequence of optimal decisions, and your sequence is composed of a finite number of steps. On the contrary, if you need to have the scientific pod orbiting around the Moon, your time horizon is (potentially) infinite, and you need to devise an infinite sequence of decisions. Moreover, because you pod is continuously orbiting, you do not have a clearly specified terminal condition from which to move backward.

When considering whether to formulate a dynamic problem in a recursive way, bear in mind that we have been able to write our intertemporal maximization problem in the form (3.3) because the payoff function and the intertemporal constraint are time-separable. [13]

In the dynamic programming jargon, the single period payoff (utility) function is often called the *return function*, the dynamic constraint is referred to as the *transition function*, while a function relating in an optimal way the control variable(s) to the state variable(s) is called the *policy function*. In our example, the policy function is $c_t^* = \varphi(k_t)$.

Before studying how an infinite horizon problem can be solved, we need to understand under which conditions the solution for problem (3.3) exists and is unique.

Stokey, Lucas, and Prescott [1989] assure us that

Theorem 1. *If: i) $\beta \in (0, 1)$, ii) the return function is continuous, bounded and strictly concave, and iii) the transition function is concave, then the maximum value function $V(k_t)$ not only exists and is unique, but it is also strictly concave, and the policy function is continuous.*

Assumption i) is usually referred to as the "discounting" hypothesis, Assumption ii) requires that the utility function $U(c_t)$ is continuous, bounded and strictly concave, while Assumption iii) constrains the production function $k_{t+1} = f(k_t) + (1 - \delta)k_t - c_t$, which must be concave for any given c_t.

The above result is neat, but it suffers from a relevant drawback: it does not allow us to work with unbounded utility functions, and hence we cannot

[13] Moreover, if our agent's utility depended upon current and future (expected) consumption levels, we would need to tackle a time inconsistency problem. For a simple introduction to this specific issue, see de la Fluente [2000, ch. 12].

assume, for example, that $U(c_t) = \ln(c_t)$. In general, it does not seem to be easy to justify an assumption that precludes utility to grow without bounds. [14]

Stokey, Lucas, and Prescott discuss the case of unbounded returns in some details (see their Theorem 4.14); here we follow Thompson [2004] in stating a more restrictive theorem that will however suffice for many applications.

Theorem 2. *(A theorem for unbounded returns). Consider the dynamic problem (3.3). Assume that* $\beta \in (0,1)$, *and that the term* $\sum_{t=0}^{\infty} \beta^t U(c_t)$ *exist and is finite for any feasible path* $\{k_t\}_{t=0}^{\infty}$, *given* k_0. *Then there is a unique solution to the dynamic optimization problem.*

Theorem 2 essentially restricts the admissible one-period payoffs to sequences that cannot grow too rapidly relatively to the discount factor.[15] To see how this theorem can be applied, consider for instance Figure 3. Clearly, when $k_0 < \bar{k}$, capital cannot become larger than $\bar{k}$. Hence, given $k_0 \in [0, \bar{k}]$, $U(c_t)$ is finite, and so is $\sum_{t=0}^{\infty} \beta^t U(c_t)$, for any feasible path $\{k_t\}_{t=0}^{\infty}$.[16]

Let us now tackle the problem of finding the solution for the Bellman equation. This is a functional equation, because we must determine the form of the unknown function $V(.)$.

Problem (3.3) tells us that, to obtain $V(k_0)$, it is necessary to maximize, with respect to c_0, the expression $U(c_0) + \beta V(k_1)$. Provided that $V(k_1)$ is continuous and differentiable – a point that we are ready to accept – the necessary condition for a maximum is obtained by differentiating $U(c_0) + \beta V(k_1)$ with respect to the current consumption, which yields:

$$U'(c_0^*) + \beta V'(k_1) \frac{\partial k_1}{\partial c_0} = 0, \tag{3.4}$$

where, of course, $\partial k_1 / \partial c_0 = -1$, from the capital accumulation equation.

The first order condition above relates the control variable (current consumption) with the current and future values of the state variable (k_0 and

[14] Unfortunately, the boundedness of the return function is an essential component of the proof for the results stated in Theorem 1. In fact, to prove existence and uniqueness for $V(k_t)$, one needs to use an appropriate fixed point theorem, because the function $V(k_t)$ is the fixed point of problem (3.3); the proofs of fixed point theorems require the boundedness of the functions involved.

[15] The idea behind the proof, that we skip, it that *if* the term $\sum_{t=0}^{\infty} \beta^t U(c_t)$ exist and is finite, then the maximum value function is finite, and we can apply an appropriate fixed-point theorem. Thompson (2004) is a very good introduction to the existence issues of the maximum value function. He also provides several result useful to characterize the maximum value function. The key reference in the literature is Stokey, Lucas, and Prescott [1989], which is however much more difficult.

[16] The conditions stated in Theorem 1 or 2 imply the fulfilment of Blackwell's sufficient conditions for a contraction.

k_1). Let us exploit the first order condition (3.4), the Bellman equation, and the constraint (1.3) to form the system:

$$\begin{cases} V(k_0) = U(c_0^*) + \beta V(k_1), \\ k_1 = f(k_0) + (1-\delta)k_0 - c_0^*, \\ U'(c_0^*) = \beta V'(k_1), \\ k_0 \; given. \end{cases} \tag{3.5}$$

The above system must determine the form of the maximum value function $V(.)$, k_1, and c_0^*, for a given k_0. Notice that the "max" operator in the Bellman equation has disappeared, simply because we have already performed this operation, via Eq. (3.4). Now consider what we often do when we deal with a functional equation: when we need to solve a difference or a differential equation, we (try to) guess the solution. Here, we can proceed in the same way. The next Section provides examples in which this strategy is successful.

4. Guess and verify

4.1. **Logarithmic preferences and Cobb-Douglas production.** In this Sub-section, we analyze a simplified version of the Brock and Mirman's (1972) optimal growth model. We shall propose a tentative solution, and then we shall verify that the guess provides the correct solution.

In this framework the consumer's time horizon is infinite, and her preferences are logarithmic:

$$W_0 = \sum_{t=0}^{\infty} \beta^t \ln(c_t), \tag{4.1}$$

with $\beta \in (0,1)$. Using the jargon, we say that we are analyzing the case of a logarithmic return function. The production function is a Cobb-Douglas characterized by a depreciation parameter as high as unity (i.e. $\delta = 1$, hence capital entirely fades away in one period). Accordingly, we have that

$$k_{t+1} = Ak_t^{\alpha} - c_t, \tag{4.2}$$

where the "total factor productivity" parameter A is a positive constant and $\alpha \in (0,1)$.

Our problem is to solve:

$$V(k_0) = \max_{c_0} \{\ln(c_0) + \beta V(k_1)\},$$

$$s.t. \; k_1 = Ak_0^{\alpha} - c_0,$$

$$k_0 \text{ given.}$$

The first order condition, corresponding to equation (3.4), is: $1/c_0^* = \beta V'(k_1)$. This condition must be exploited in the problem above to substitute out consumption. This yields:

$$\left\{ \begin{array}{c} V(k_0) = \ln([\beta V'(k_1)]^{-1}) + \beta V(k_1) \\ k_1 = Ak_0^\alpha - \frac{1}{\beta V'(k_1)} \\ k_0 \text{ given.} \end{array} \right. \tag{4.3}$$

This formulation makes it apparent once again that the Bellman equation is a functional equation: it involves the function $V(.)$ and its derivative $V'(.)$; the constraint incorporates the initial condition for the state variable. In fact, once $V'(.)$ is known, because k_0 is given, the constraint determines k_1, and therefore the evolution for the capital stock. Notice that system (4.3) corresponds to (3.5), but for the fact that here we have directly substituted out c_0^* thanks to the explicit formulation $c_0^* = [\beta V'(k_1)]^{-1}$.

Our attack against the functional equation is conducted by means of a tentative solution, which in this case takes the form:

$$V(k_t) = e + f\ln(k_t), \tag{4.4}$$

where e and f are two constants to be determined, i.e. two undetermined coefficients. Spend a few seconds in considering the guess (4.4). It is a linear transformation of the utility function, where the control variable has been substituted by the state variable. When trying to find a tentative solution, it is usually sensible to proceed in this way, i.e. to start with a guess that is similar to the return function.

Taking the guess seriously, and exploiting it in (4.3), we obtain, for a given k_0:

$$\left\{ \begin{array}{c} e + f\ln(k_0) = \ln\left(\frac{k_1}{\beta f}\right) + \beta[e + f\ln(k_1)] \\ k_1 = Ak_0^\alpha - \frac{k_1}{\beta f} \end{array} \right. . \tag{4.5}$$

One immediately notices that the second equation may be solved for k_1:

$$k_1 = \frac{\beta f}{1+\beta f}Ak_0^\alpha. \tag{4.6}$$

Substituting the above result in the first equation in (4.5) gives:

$$e + f\ln(k_0) = \ln\left(\frac{1}{1+\beta f}Ak_0^\alpha\right) + \beta\left[e + f\ln\left(\frac{\beta f}{1+\beta f}Ak_0^\alpha\right)\right].$$

Exploiting the usual properties of logarithmic functions, we obtain:

$$e+f\ln(k_0)=\ln\left(\frac{1}{1+\beta f}\right)+\ln(Ak_0^\alpha)+\beta e+\beta f\ln\left(\frac{\beta f}{1+\beta f}\right)+\beta f\ln(Ak_0^\alpha),$$

or:

$$\begin{aligned} e+f\ln(k_0) &= -\ln(1+\beta f)+ \\ &+\ln A+\alpha\ln k_0+\beta e+\beta f\ln(\beta f)-\beta f\ln(1+\beta f)+\beta f\ln A+\alpha\beta f\ln k_0. \end{aligned}$$

The above equation must be satisfied for *any* k_0 and for *any* admissible value of the parameters A,β, and α. Hence, it must be true that:

$$\begin{cases} f=\alpha+\alpha\beta f \\ e=-\ln(1+\beta f)+\ln A+\beta e+\beta f\ln(\beta f)-\beta f\ln(1+\beta f)+\beta f\ln A \end{cases}.$$

From the first equation in the system above we obtain:

$$f=\frac{\alpha}{1-\alpha\beta}.$$

Notice that $f>0$, because $\alpha,\beta\in(0,1)$. This implies $V'(k)>0$, a sensible result which tells us that a richer consumer enjoys a higher overall utility. Substituting f in the second equation gives:

$$e=\frac{1}{1-\beta}\left[\frac{\alpha\beta}{1-\alpha\beta}\ln(\alpha\beta)+\ln(1-\alpha\beta)+\frac{1}{1-\alpha\beta}\ln A\right].$$

The equation above provides us with some intuition concerning the reason why we need the assumption $\beta<1$: were this requirement not fulfilled, the maximum value function would "explode" to infinity.

Because f and e are independent of capital, our guess (4.4) is verified.

Substituting f into (4.6), we obtain: $k_1=\alpha\beta Ak_0^\alpha$, which is the specific form of the function $k_{t+1}=\phi(k_t)$ introduced in Section 3.

From the first order condition, we know that $c_0^*=1/\left[\beta V'(k_1)\right]$, hence, using our guess, the computed value for f, and the fact that $k_1=\alpha\beta Ak_0^\alpha$, we obtain: $c_0^*=(1-\alpha\beta)Ak_0^\alpha$, which is the form of the function $c_t=\varphi(k_t)$ in this example.

The consumption function $c_0^*=(1-\alpha\beta)Ak_0^\alpha$ is a neat but somewhat economically uninteresting result: it prescribes that the representative agent must consume, in each period, a constant share $(1-\alpha\beta)$ of her current income, Ak_0^α.

Notice that our consumption function relates the control variable to the state variable: hence, it is the policy function.

From $k_1=\alpha\beta Ak_0^\alpha$, we can easily obtain the steady state level for capital, that is: $\hat{k}=(\alpha\beta A)^{\frac{1}{1-\alpha}}$.

Having already obtained the maximum value function, it is certainly funny to ask ourselves to prove that it exits and is unique. Nevertheless, this is exactly what we are going to do in the next five paragraphs. The reason is that we wish to develop a line of reasoning that may be helpful also when it is not possible to find a closed-form solution for the maximum value function. The uninterested reader may skip these paragraphs. In this case, however, Exercise 4 will prove rather difficult.

Recall that $U(c_t) = \ln(c_t)$, and that $k_{t+1} = Ak_t^\alpha - c_t$. Notice that the path for capital characterized by the fastest possible growth for capital itself is obtained by choosing zero consumption at each period of time. This path is given by $k_1 = Ak_0^\alpha$, $k_2 = Ak_1^\alpha$, ... Hence, in general, we have that

$$\begin{aligned}\ln(k_{t+1}) &= \ln(A) + \alpha \ln(k_t) = \ln(A) + \alpha \ln(A) + \alpha^2 \ln(k_{t-1}) = \\ &= \ ... \ = \frac{(1-\alpha^{t+1})}{1-\alpha} \ln(A) + \alpha^{t+1} \ln(k_0).\end{aligned}$$

Notice also that the largest one-period utility is obtained by consuming the entire output that the representative agent can produce in that period, i.e.

$$U(c_t) = \ln(Ak_t^\alpha) = \ln(A) + \alpha \ln(k_t).$$

Therefore, if we follow the policy prescribing to save everything up to period t and then to consume the entire output, we obtain:

$$U(c_t) = \ln(A) + \alpha \ln(k_t) = \frac{1-\alpha^{t+1}}{1-\alpha} \ln(A) + \alpha^{t+1} \ln(k_0).$$

Imagine, counterfactually, that the above policy could be followed in *every* period. In this case, the lifetime utility for the representative agent would be:

$$\begin{aligned}\sum_{t=0}^{\infty} \beta^t \ln(c_t) &= \sum_{t=0}^{\infty} \beta^t \left[\frac{1-\alpha^{t+1}}{1-\alpha} \ln(A) + \alpha^{t+1} \ln(k_0) \right] = \\ &= \frac{1}{(1-\beta)(1-\beta\alpha)} \ln(A) + \frac{\alpha}{1-\beta\alpha} \ln(k_0).\end{aligned}$$

It is obvious that the above expression is finite. Clearly, any *feasible* path would yield a lower lifetime utility, therefore, any feasible sequence of payoffs must be bounded (in present value), and this implies that the maximum value function must also be bounded. Because $\sum_{t=0}^{\infty} \beta^t U(c_t)$ exist and is finite for any feasible path, Theorem 2 applies and the maximum value function is unique**.**

Exercise 1. *Assume that i) the single period utility function is:* $\ln(c_t) + \gamma\ln(1-l_t)$, *where* $l_t \in [0,1]$ *is the share of time devoted to labour, and* $\gamma > 0$; *ii) the dynamic constraint is* $k_{t+1} = Ak_t^{\alpha} l_t^{(1-\alpha)} - c_t$, *where* $A > 0$ *and* $0 < \alpha < 1$. *Find the value function, and the related policy functions.*

(Hint: because we have two control variables, the first order conditions ....)

Exercise 2. *(Habit persistence) Assume that i) the single period utility function is:* $\ln(c_t) + \gamma\ln(c_{t-1})$, *where* $\gamma > 0$; *ii) the dynamic constraint is* $k_{t+1} = Ak_t^{\alpha} - c_t$, *where* $A > 0$ *and* $0 < \alpha < 1$. *Find the value function, and the related policy function.*

(Hint: consider past consumption as a state of the system, hence the value function has two arguments: k_t *and* c_{t-1}*....)*

Exercise 3. *Assume that i) the single period utility function is:* $c_t^{1-\gamma}/(1-\gamma)$, *where* $\gamma \in [0,1)\cup(1,\infty)$; *ii) the dynamic constraint is* $k_{t+1} = Ak_t - c_t$, *where* $A > 0$. *Find the value function, and the related policy function.*

Exercise 4. *Provide – for the return and the transition functions used in the previous exercise – a condition on* γ *such that the boundedness conditions in Theorem 2 is satisfied.*

4.2. **Quadratic preferences with a linear constraint.** We now consider a consumer, whose time horizon is again infinite, characterized by "quadratic" preferences:

$$W_0 = \sum_{t=0}^{\infty} \beta^t \left(\varepsilon + \gamma c_t - \frac{\eta}{2} c_t^2\right), \text{ with } \beta \in (0,1). \tag{4.7}$$

Because the parameters γ and η are assumed to be positive, the marginal utility for our consumer is positive for $c_t < \gamma/\eta$, while it is negative for $c_t > \gamma/\eta$. Hence, the single-period utility is maximum when $c_t = \gamma/\eta$, which is called the "bliss point" in consumption (the corresponding utility is $\varepsilon + \gamma^2/(2\eta)$).[17] The intertemporal constraint is linear in the state variable k_t:

$$k_{t+1} = (1+r)k_t - c_t. \tag{4.8}$$

We may interpret k_t as the consumer's financial assets and r as the interest rate. Notice that, because the utility function is strictly concave and bounded and the transition function is concave, we can be sure that the value function is unique and strictly concave (Theorem 1).

[17] We do not take a position about ε: it can be positive, negative or nought. For example, a negative ε implies that our consumer needs a minimal amount of consumption to start enjoying life (and hence having a positive utility).

Our problem is to solve:

$$V(k_0) = \max_{c_0} \left\{ \varepsilon + \gamma c_0 - \frac{\eta}{2} c_0^2 + \beta V(k_1) \right\},$$
$$s.t.\ k_1 = (1+r)k_0 - c_0,$$
$$k_0 \text{ given.}$$

As before, we find the first order condition, which is: $\gamma - \eta c_0^* = \beta V'(k_1)$. From the dynamic constraint we obtain $c_0 = (1+r)k_0 - k_1$, which is used to substitute consumption out of the Bellman equation and out of the first order condition. This yields

$$(4.9) \qquad \begin{cases} V(k_0) = \varepsilon + \gamma[(1+r)k_0 - k_1] - \frac{\eta}{2}[(1+r)k_0 - k_1]^2 + \beta V(k_1) \\ \gamma - \eta[(1+r)k_0 - k_1] = \beta V'(k_1), \\ k_0 \text{ given.} \end{cases}$$

Notice that system (4.9) corresponds to (3.5), but for the fact that we have substituted out c_0^* exploiting the linear constraint (4.8).

The logic of the solution method is the same we have experienced in the previous example: accordingly, we now introduce the tentative solution, which takes the same functional form of the return function:

$$(4.10) \qquad V(k_t) = g + hk_t + \frac{m}{2}k_t^2,$$

where g, h, and m are the undetermined coefficients. The (small) difference with the example in Sub-section 4.1 is that in this case we shall set up a three equations system, because we need to determine three coefficients.

From the second equation in (4.9), we obtain:

$$k_0 = \frac{1}{1+r}\left[\frac{\gamma - \beta h}{\eta} + \frac{\eta - \beta m}{\eta} k_1\right].$$

The equation above grants us that – in the present example – the function $k_{t+1} = \phi(k_t)$ is linear.

Substituting the guess (4.10) for $V(k_0)$, and $V(k_1)$ in the first equation in (4.9), and exploiting the above expression, we obtain a quadratic equation in k_1. Because this equation must be satisfied for any value of k_1, it must be true that:

(4.11)
$$\begin{cases} g + \frac{h}{1+r}\left(\frac{\gamma-\beta h}{\eta}\right) + \frac{m}{2(1+r)^2}\left(\frac{\gamma-\beta h}{\eta}\right)^2 = \varepsilon + \frac{\gamma}{\eta}(\gamma - \beta h) - \frac{(\gamma-\beta h)^2}{2\eta} + \beta g \\ \frac{h}{1+r}\left(\frac{\eta-\beta m}{\eta}\right) + \frac{m}{(1+r)^2}\left(\frac{\eta-\beta m}{\eta}\right)\left(\frac{\gamma-\beta h}{\eta}\right) = -\frac{\gamma\beta m}{\eta} + \frac{(\gamma-\beta h)\beta m}{\eta} + \beta h \\ \frac{m}{2(1+r)^2}\left(\frac{\eta-\beta m}{\eta}\right)^2 = -\frac{(\beta m)^2}{2\eta} + \frac{\beta m}{2} \end{cases}.$$

Inspection of the above system reveals that it is convenient to solve first the third equation: the unique unknown it involves is m. From this equation, we immediately obtain:

$$m = \frac{\eta}{\beta}[1 - \beta(1+r)^2]. \tag{4.12}$$

Inserting (4.12) into the second equation in (4.11) gives an equation which can be solved for h, yielding:

$$h = \frac{\gamma}{\beta}\frac{\left[\beta(1+r)^2 - 1\right]}{r}. \tag{4.13}$$

Finally, we can use (4.13) and (4.12) into the first equation in (4.11), to obtain g:

$$g = \frac{\epsilon}{1-\beta} + \frac{\gamma^2}{2\eta\beta(1-\beta)}\left[\frac{1-\beta(1+r)}{r}\right]^2. \tag{4.14}$$

From (4.14) we see once again why the assumption $\beta < 1$ is crucial: it prevents the maximum value function from exploding to infinity. Because g, h and m are independent of the state variable, that is capital, our guess (4.10) is correct. [18]

4.2.1. *A particular case.* Our discussion in Sub-section 1.2 tells us that an optimal consumption path must satisfy the Euler equation. Hence, the period t marginal utility for consumption is equal to the next period marginal utility, discounted by β and capitalized by means of the net marginal productivity of capital (refer to equation (1.9)). In the present framework, the interest rate r plays the role of the net marginal productivity of capital, $f'(k_{t+1}) - \delta$. Notice that equalizing the marginal productivity of capital to the interest rate, we assume that the goods market is competitive: only in this case capital is paid its marginal productivity (net of depreciation).

Because the dynamic constraint is linear (refer to (4.8)), $f'(k_{t+1})$ must not change over time; this can happen with a potentially varying capital stock only in two cases. Either $f(k_t)$ is linear, or $f'(k_{t+1})$ stays constant because the capital stock actually does not change over time, which means that it is in its steady state. Linearity of the production function is a strong assumption because it amounts to accept that the marginal productivity of capital is constant. Hence, we focus on the latter case, assuming that capital is constant at its long run value, $\hat{k}$. Notice, that, for this level of the

[18]Our example has required a good deal of calculations. When a linear-quadratic model involves two (or more) state variables, it has to be solved by means of numerical techniques involving matrix manipulations. Refer to Chow [1997] or to Sargent and Ljungqvist [2004].

capital stock, it must be true that $\beta\left[f'(\hat{k})+(1-\delta)\right]=1$ (equation (1.10)), and hence that $\beta(1+r)=1$. In the steady state, the impatience parameter β exactly offsets the positive effects on saving exerted by its reward.

Notice that "around" the steady state, the value function parameters (4.12-4.14) simplify to:

$$\begin{cases} m=-\frac{\eta}{\beta}r \\ h=\frac{\gamma}{\beta} \\ g=\frac{\epsilon}{1-\beta} \end{cases}.$$

In this particular−but sensible−case, it is easy to find the policy function. Exploiting the tentative solution (4.10), and the parameters computed above, the second equation in (4.9) becomes: $\gamma-\eta[(1+r)k_0-k_1]=\gamma-\eta r k_1$, which confirms $k_1=k_0$. Hence, from the dynamic constraint (4.8), we get: $c_0^*=rk_0$. It is also immediate to check that the transversality condition holds. In fact, in this model the tvc is:

$$\lim_{t\to\infty}\beta^t(\gamma-\eta c_0^*)k_0,$$

which converges to 0 simply because $\lim_{t\to\infty}\beta^t=0$.

Exercise 5. *(a) Study the dynamics for* c_t *and* k_t *if* $\beta(1+r)>1$. *Discuss whether the transversality condition is always satisfied. (b) Study the dynamics for* c_t *and* k_t *if* $\beta(1+r)<1$. *Discuss whether the transversality condition is always satisfied.*

4.2.2. *An application to the growth model.* Let us now consider again the growth model with logarithmic preferences given by (4.1). Assume that the production function is Cobb-Douglas; but now take account of the fact that capital depreciates slowly. Hence, our intertemporal constraint becomes:

$$k_{t+1}=Ak_t^{\alpha}+(1-\delta)k_t-c_t. \tag{4.15}$$

Because there is no way to obtain a closed form analytic solution, what we can do is to use a linear-quadratic approximation of our model. This, of course, implies the need to choose a point around which to approximate. The standard choice for this point is the steady state, which is certainly sensible if our research project involves, for example, the introduction of productivity shocks and the study of their effects in a "mature" economic system.

Hence, we determine first the steady state. With the production function (4.15), the steady state equations (2.8) become:

$$\begin{cases} \beta[\alpha A\hat{k}^{\alpha-1}+(1-\delta)]=1 \\ \hat{c}=A\hat{k}^{\alpha}-\delta\hat{k} \end{cases}. \tag{4.16}$$

The above system allows to determine the consumption and capital steady-state levels.[19] We now apply Taylor's theorem to the logarithmic utility function, obtaining:

$$\ln(c_t) = \ln(\hat{c}) + \frac{1}{\hat{c}}(c_t - \hat{c}) - \frac{1}{2\hat{c}^2}(c_t - \hat{c})^2. \tag{4.17}$$

As for the capital accumulation constraint, we truncate the Taylor's approximation to the first term, which yields:

$$k_{t+1} = A\hat{k}^{\alpha} + \alpha A\hat{k}^{\alpha-1}(k_t - \hat{k}) + (1-\delta)\hat{k} + (1-\delta)(k_t - \hat{k}) - \hat{c} - (c_t - \hat{c}),$$

which immediately becomes:

$$k_{t+1} = A\hat{k}^{\alpha} + (1-\delta)\hat{k} - \hat{c} + \left[\alpha A\hat{k}^{\alpha-1} + (1-\delta)\right](k_t - \hat{k}) - (c_t - \hat{c}),$$

and hence, using (4.16):

$$k_{t+1} - \hat{k} = \frac{1}{\beta}(k_t - \hat{k}) - (c_t - \hat{c}). \tag{4.18}$$

Equations (4.17) and (4.18) lead to the very same structure that can be found in (4.7) and (4.8), and hence we can solve the approximate problem using the tentative solution postulated for the linear-quadratic problem (the relevant variables are the deviations of capital and consumption from the steady state).

5. Two useful results

The "guess and verify" technique is useful only when a closed form solution exists. Unfortunately, only a few functional forms for the payoff function and for the dynamic constraint allow for a closed form maximum value function: the previous Section almost works out the list of problems allowing for a closed form maximum value functions. Hence, we very often need to "qualify" the solution, identifying some of its characteristics or properties, without solving the model. In this Section, we review two important results, that may be helpful in studying the solution for a Bellman equation.

5.1. **The Envelope Theorem.** We now apply the envelope theorem to the standard growth model, as formulated in Problem (3.3). In other words, we concentrate – for simplicity – on a specific application of the envelope theorem. However, the results that we obtain, besides being important, are of general relevance.

---

[19]Which are: $\hat{k} = \left(\frac{\alpha\beta A}{1-\beta(1-\delta)}\right)^{\frac{1}{1-\alpha}}$, and $\hat{c} = \left(\frac{1-\beta[1-\delta(1-\alpha)]}{\alpha\beta}\right)\left(\frac{\alpha\beta A}{1-\beta(1-\delta)}\right)^{\frac{1}{1-\alpha}}$.

To save on notation, we now denote the dynamic constraint by $k_1 = g(k_0, c_0)$, accordingly, the dynamic programming formulation for our utility-maximization problem becomes:

$$V(k_0) = \max_{c_0} \{U(c_0) + \beta V(k_1)\},$$
$$s.t.\ k_1 = g(k_0, c_0),\ k_0 \text{ given.}$$

We already know that the first order condition with respect to the control variable is $U'(c_0^*) + \beta V'(k_1)\frac{\partial k_1}{\partial c_0} = 0$. Consider now the Bellman problem above, assuming to be on the optimal path. In this case, we have $V(k_0) = U(c_0^*) + \beta V(k_1)$ (the max operator disappears exactly because we already are on the path in which consumption is optimal). The total differential for the last equation is: $dV(k_0) = dU(c_0^*) + \beta dV(k_1)$, or: $V'(k_0)dk_0 = U'(c_0^*)dc_0 + \beta V'(k_1)dk_1$. The differential for k_1 can be easily obtained from the dynamic constraint: $dk_1 = g_k(k_0, c_0)dk_0 + g_c(k_0, c_0)dc_0$. [20]

Exploiting dk_1, the total differential for the Bellman equation becomes:

$$V'(k_0)dk_0 = U'(c_0^*)dc_0 + \beta V'(k_1)\left[g_k(k_0, c_0)dk_0 + g_c(k_0, c_0)dc_0\right].$$

This expression, using the first order condition for c_0, reduces to:

$$V'(k_0) = \beta V'(k_1)g_k(k_0, c_0).$$

This is an application of the Envelope theorem: we have simplified the total differential precisely because we are on the optimal path, and hence the first order condition must apply.

Notice that we could have expressed the above result as follows:

$$V'(k_0) = \beta V'(k_1)\frac{\partial k_1}{\partial k_0}, \tag{5.1}$$

where $\partial k_1/\partial k_0$ is the partial derivative, $g_k(k_0, c_0)$.

Equation (5.1) can be useful in several contexts. In fact, it can be reformulated in a very convenient way. Because the first order condition states that: $U'(c_0^*) = \beta V'(k_1)$, it must also be true that $U'(c_1^*) = \beta V'(k_2)$. With these facts in mind, we forward once equation (5.1), and we obtain:

$$U'(c_0^*) = \beta U'(c_1^*)g_k(k_1, c_1^*), \tag{5.2}$$

which is the Euler equation (bear in mind that, in the growth example, $g_k(k_1, c_1) = f'(k_1) + (1-\delta)$, and refer to (1.9)). Not only it is often easier

[20] We denote by a subscript the partial derivatives. Accordingly, $g_k(k_0, c_0)$ is the partial derivative of $g(k_0, c_0)$ with respect to capital, and so on. This convention shall be adopted whenever we need to differentiate a function with two or more arguments.

to solve numerically equation (5.2) than Bellman's one – as we shall see in Sub-section 6.2 – but equation (5.2) can be interpreted without referring to the still unknown maximum value function.

Notice that Equation (5.2) corresponds exactly to Equation (1.9): this reassures us about the fact that the Bellman's approach and the Lagrange's one lead to the same result.

5.2. **The Benveniste and Scheinkman formula.** In this Sub-section – that can be skipped during the first reading – we consider a more general framework, where the return function does not depend only on the control variables, but it also depends on the state. In this case the dynamic programming formulation is:

$$V(k_0) = \max_{c_0} \left\{ Q(k_0, c_0) + \beta V(k_1) \right\},$$
$$s.t.\ k_1 = g(k_0, c_0),\ \ k_0 \text{ given.}$$

The first order condition with respect to the control variable yields:

$$Q_c(k_0, c_0^*) + \beta V_k(k_1) \frac{dk_1}{dc_0} = 0.$$

Assume, as in the previous Sub-section, to be on the optimal path, so that: $V(k_0) = Q(k_0, c_0^*) + \beta V(k_1)$. The total differential for the last equation is: $dV(k_0) = dQ(k_0, c_0^*) + \beta dV(k_1)$, or:

$$V_k(k_0)dk_0 = Q_c(k_0, c_0^*)dc_0 + Q_k(k_0, c_0^*)dk_0 + \beta V_k(k_1)dk_1.$$

The differential for the first period capital is obtained from the dynamic constraint, and is $dk_1 = g_k(k_0, c_0)dk_0 + g_c(k_0, c_0)dc_0$. Hence, the total differential for the Bellman's equation becomes:

$$\begin{aligned} &V_k(k_0)dk_0 = \\ &= Q_c(k_0, c_0^*)dc_0 + Q_k(k_0, c_0^*)dk_0 + \beta V_k(k_1) \left[g_k(k_0, c_0^*)dk_0 + g_c(k_0, c_0^*)dc_0 \right]. \end{aligned}$$

Using the first order condition, the equation above reduces to:

$$V_k(k_0) = Q_k(k_0, c_0^*) + \beta V_k(k_1) g_k(k_0, c_0^*). \tag{5.3}$$

This is the Benveniste and Scheinkman formula, which can be obtained as an application of the Envelope theorem.

The usefulness of the Benveniste and Scheinkman formula (5.3) can be appreciated considering that, in many problems, there is not a unique way to define states and controls. For example, in the growth model, one could define as control variable not consumption, but gross savings (which means that the control is $s_t = f(k_t) - c_t$). When depreciation is complete, this modification implies that the next-period state is equal to the current control ($k_{t+1} = s_t$), and the Bellman problem (3.3) with $\delta = 1$, becomes:

$$V(k_0) = \max_{s_0} \{Q[f(k_0) - s_0] + \beta V(k_1)\}, \quad s.t.\ k_1 = s_0, \quad k_0 \text{ given.} \tag{5.4}$$

With this formulation, the first order condition is: $U'[f(k_0) - s_0^*] = \beta V'(k_1)$.

If we differentiate the Bellman equation "on the optimal path", we obtain:

$$V'(k_0)dk_0 = Q'[f(k_0) - s_0^*]\,[df(k_0) - ds_0] + \beta V'(k_1)dk_1.$$

Because $dk_1/ds_0 = 1$, using the first order condition the differential can be reduced to:

$$V'(k_0) = Q'[f(k_0) - s_0^*]f'(k_0), \tag{5.5}$$

which is the Benveniste and Scheinkman formula in our context.

Formula (5.5) is interesting because it shows that the Benveniste and Scheinkman result can be used to highlight a relation between the initial period maximum value function, the return function and the dynamic constraint. The Benveniste and Scheinkmann formula leads to such relation *when the partial derivative of the dynamic constraint with respect to the current state is 0.* For this to be true, it is necessary that the initial period state variable (k_0) is excluded from the dynamic constraint, as it happens in problem (5.4). The example we have just developed shows that this can be achieved by means of a proper variable redefinition (see Sargent [1987] pp. 21-26 for a discussion and an alternative example).

Exercise 6. *Apply the Benveniste-Scheinkman formula to the standard growth model, and show that the maximum value function is concave.*

Exercise 7. *By means of an appropriate variable redefinition, show that the Benveniste-Scheinkman formula (5.5) applies to the standard growth model when $\delta < 1$.*

6. A "paper and pencil" introduction to numerical techniques

As already underscored, the "guess and verify" technique is useful only in the few cases in which a closed form solution for the maximum value function exists. In this Section, we illustrate two alternative techniques that can be used to approximate numerically the maximum value function: the value function iteration method and the (more up-to-date) collocation technique. We do this by means of simple examples, which rely on the elementary version of the Brock and Mirman model solved in Sub-section 4.1.

6.1. **Value function iteration based on the discretization of the state space.** Let us consider again the version of Brock and Mirman model that we faced in Sub-section 4.1, assuming, however, that we are not able to find the explicit solution for this problem, so that we need to compute a numerical approximation for the solution. The reason why we solve a well-understood problem by means of a numerical technique is to allow for the comparison of the approximated solution we obtain with the exact one, that we already know.

Because we are moving in the direction of using numerical techniques, the first thing we need to do is to pick the values we want to assign to our parameters. This is readily done. A sensible value for α is 0.3. In fact, α represents the capital income share of output, a value which is between 0.25 and 0.33 for most OECD countries. As for β we choose 0.97: we know from equation (1.10) that – in the steady state – β is equal to the reciprocal of the marginal productivity of capital, which represents also the interest factor. Hence, if a period represents a year, $\beta = 0.97$ implies a long-run annual (real) interest factor approximately equal to 1.03, a realistic value. As for the total factor productivity parameter A, we choose a value such that the long-run capital level, $\hat{k} = (\alpha\beta A)^{\frac{1}{1-\alpha}}$, is unity. Hence, $A \simeq 3.43643$. This is a normalization: we have decided to measure output using a reference unit such that the capital long-run value is exactly equal to one.

Accordingly, our problem is to solve:

$$\begin{gathered} V(k_0) = \max_{c_0} \left\{ \ln(c_0) + 0.97V(k_1) \right\}, \\ s.t.\ k_1 = 3.43643k_0^{0.3} - c_0, \\ k_0 \text{ given.} \end{gathered} \tag{6.1}$$

We assume to be interested in solving the problem for capital values that are around the steady state.[21]

The state variable is continuous, nevertheless we now consider it as if it were discrete: it is this approximation that allows to use the numerical technique we are describing. To fix the ideas, we consider only five possible levels for capital, which are $\{0.98, 0.99, 1, 1.01, 1.02\}$.

[21]Justify this hypothesis when you have read and understood this Sub-section.

Notice that, if we knew the function $V(k_1)$, it would be relatively easy to solve problem (6.1): the solution would be the consumption level c_0^* that allows to obtain the highest $V(k_0)$, provided that $k_0, k_1 \in \{0.98, 0.99, 1, 1.01, 1.02\}$. For example, assume that $V(k_1) = 20k_1$ (this choice is completely arbitrary; actually we know that it is wrong: we just want to illustrate what we mean when we say that – knowing $V(k_1)$ – it is easy to solve problem (6.1)). When $k_0 = 0.98$, the problem becomes:

$$
\begin{aligned}
V(k_0 = 0.98) &= \max_{c_0} \{\ln(c_0) + 0.97 \times 20k_1\}, \\
s.t.\ k_1 &= 3.43643(0.98)^{0.3} - c_0, \\
k_0 &= 0.98.
\end{aligned}
\tag{6.2}
$$

To tackle the problem, we must leave our standard tool – the derivative – on the shelf, because our problem is not continuous: we have decided to compute $V(k_1)$ only in a few points. Hence, we need to perform all the required calculations. First, we express the constraint as: $c_0 = 3.43643(0.98)^{0.3} - k_1 = 3.41566 - k_1$, and we compute the consumption levels that allow k_1 to take one of the five feasible values. These are:

k_1	0.98	0.99	1	1.01	1.02
c_0	2.43566	2.42566	2.41566	2.40566	2.39566

Second, we consider the postulated maximum value function: for $V(k_1) = 20k_1$ the right hand side of problem (6.2) takes, in correspondence of the five couples $\{k_1, c_0\}$ computed above, the following values:

$r.h.s.$(6.2)	19.90222	20.09210	20.28197	20.47182	20.66166

According to the above calculations, the highest value is obtained for the consumption choice $c_0 = 2.39566$, hence $V(k_0 = 0.98)$ takes the value 20.66166.

This calculation should be repeated for every remaining $k_0 \in \{0.99, 1, 1.01, 1.02\}$, this would give us the period 0 maximum value function for each state.

However, there is no need to perform all these calculations: the point we wish to underscore here is that the few calculations we have presented above are already enough to conclude that the postulated function $V(k_1) = 20k_1$, is incorrect. From our analysis in Section 3, we know that – in infinite horizon models – the maximum value function is time independent. Accordingly, the functional form we have assumed for period 1 must apply also to period 0. Hence, we should expect to find that $V(k_0 = 0.98) = 19.6$, which is not the case.

While the usefulness of the above example lies in the fact that we have understood which kind of computations are required, we must now tackle the real issue: we need to know how to obtain the unknown maximum value function.

The strategy to get it prescribes to:

- (Step 1) attribute a set of arbitrary values to $V(k_1)$. (Denote this initial set of values by $V_0(k_1)$);
- (Step 2) solve the problem (6.1) for each state k_0, finding a new (and hence different) maximum value function (which, of course, substantiates in a new set of values, denoted by $V_1(k_0)$);
- (Step 3) obtain $V_2(k_0)$, using $V_1(k_1) = V_1(k_0)$ [22] as a new initial guess for the value function;
- (Step 4) iterate the step above until $V_n(k_0)$ and $V_{n+1}(k_0)$ are "sufficiently close".

The convergence of the above procedure to a set of values that represent the true maximum value function is guaranteed whenever the hypothesis in Theorem 1 or in Theorem 2 are satisfied. In these cases, successive iterations of the value function converge to the true value function, and this convergence takes place for any starting point, i.e. for any initial arbitrary value function.

We now provide some further details about the above procedure by means of our extremely simplified example.

We start by choosing a set of values for $V(k_1)$. A commonly chosen set of initial values is $V_0(k_1) = 0$ for any k_1. This choice corresponds to Step 1 in our procedure.

As for Step 2, we start by noticing that the above choice implies that we need to face problems of the following type:

$$V(k_0) = \max_{c_0} \left\{ \ln(c_0) \right\},$$
$$s.t.\ k_1 = 3.43643 k_0^{0.3} - c_0,$$
$$k_0 \text{ given.}$$

Notice that, as before, both k_0 and k_1 must take one of the possible values for capital (i.e. $\{0.98, 0.99, 1, 1.01, 1.02\}$): this restrict the set of feasible consumption levels. For example, if $k_0 = 0.98$, for $k_1 = \{0.98, 0.99, 1, 1.01, 1.02\}$, we compute, as before: $c_0 = \{2.43566, 2.42566, 2.41566, 2.40566, 2.39566\}$; the corresponding utilities are $\{0.89022, 0.88610, 0.88197, 0.87782,$

[22] If you feel confused by this, just go on reading.

$0.87366\}$. The highest utility level is reached for $c_0 = 2.43566$, which is the consumption level implying that the next period capital is $k_1 = 0.98$.

Hence, the solution for this specific problem is $V_1(k_0 = 0.98) = 0.89022$.

Repeating this reasoning we obtain: $V_1(k_0 = 0.99) = 0.89449$, $V_1(k_0 = 1) = 0.89871$, $V_1(k_0 = 1.01) = 0.90288$, $V_1(k_0 = 1.02) = 0.90701$. In every problem, the solution corresponds to the maximum feasible consumption level, which is the one corresponding to $k_1 = 0.98$. In other words, for any $k_0 \in \{0.98,\ 0.99,\ 1,\ 1.01,\ 1.02\}$, c_0 is such that $k_1 = 0.98$. This is hardly surprising: since we have arbitrarily chosen $V_0(k_1) = 0$, capital bears no future value, and hence it is sensible to choose to consume as much as possible.[23]

The key point is that our maximization procedure has led to a set of values for $V_1(k_0)$. Accordingly, we have completed Step 2 in the procedure.

Hence – Step 3 – we use these values as a new guess for the maximum value function. This bit of the procedure is inspired by the fact that the true maximum value function is time independent: at different dates it must take the same values for any capital stock. Accordingly, we now assume $V_1(k_1) = \{0.89022,\ 0.89449,\ 0.89871,\ 0.90288,\ 0.90701\}$.

In this second iteration for the maximization problem, when $k_0 = 0.98$, consumption may again take the values $c_0 = \{2.43566,\ 2.42566,\ 2.41566,\ 2.40566,\ 2.39566\}$. When c_0 is 2.43566, (that is, when it takes the value guaranteeing $k_1 = 0.98$), the corresponding "overall utility" is $\ln(2.43566)$ plus β times $V_1(k_1 = 0.98) = 0.89022$ (which gives $V_2(k_0 = 0.98) = 1.75373$). In words, this is the "overall value" obtained by choosing the consumption level that allows to transfer the current capital level to the next period.

Considering the whole set of choices, $c_0 = \{2.43566,\ 2.42566,\ 2.41566,\ 2.40566,\ 2.39566\}$, we find that the values associated are $\{\ln(2.43566) + 0.97 \times 0.89022,\ \ln(2.42566) + 0.97 \times 0.89449,\ \ln(2.41566) + 0.97 \times 0.89871,\ \ln(2.40566) + 0.97 \times 0.90288,\ \ln(2.39566) + 0.97 \times 0.90701\}$, which are $\{1.75373,\ 1.75376,\ 1.75372,\ 1.75362,\ 1.75346\}$.

Notice that the second value in the row, 1.75376 is the highest one. Hence, the optimal choice is now to opt for a consumption level such that the next period capital is 0.99. In words, if the initial capital is 0.98 and if we take account of the future period (discounted) utility, it is optimal to save 0.01 units of capital.

When we repeat this reasoning for the other capital values in our grid, we obtain that: if $k_0 = 0.99$, the consumption optimal choice is such that k_1 remains at 0.99; when $k_0 = 1$, the consumption optimal choice is such that k_1 is again 0.99; when $k_0 = 1.01$ or 1.02, the consumption optimal choice is such

[23]Here the logic is the same we used in Sub-section 1.3 to argue that $k_{T+1} = 0$.

that $k_1 = 1$. In correspondence to the optimal choices described above, the value function is: $V_2(k_1) = \{1.75376, 1.75804, 1.76228, 1.76649, 1.77065\}$. The obtaining of this set of values concludes Step 3 in our procedure.

Of course, a normal person is already fed up with all these calculations. This is the point to let our computer do the calculations, hence performing Step 4. It is easy to write a program using Matlab (or Gauss) that carries out this task.[24] The results of this program are reported in Table 1.

[Insert Table 1]

In the simulation, the convergence criterion we have chosen is as follows: we let the computer calculate−for each gridpoint−the difference between the final and the initial value of $V(k_0)$ (i.e. $V_{n+1}(k_0) - V_n(k_0)$); then we considered the absolute values of these differences and we let the routine to pick the largest. When this is below $10^{(-5)}$, we let the computer to stop. On a 3 Ghz Pentium computer, this happens after about 30/100 of a second, at the 376^{th} iteration.

Because the computed value function converges to the true maximum value function, when the set of values representing $V_{n+1}(k_0)$ is "almost identical" to the one used to represent $V_n(k_0)$, they are also "almost identical" to the true set of values, which is to the true $V(k_0)$.

Having chosen an exercise for which we know the exact solution, we have been able to provide, as a reference point, the true values for $V(k_1)$.

Exercise 8. *Check that the values for $V(k)$ in the last line of Table 1 are correct.*

From Table 1, we can see that the values for $V(k_1)$ smoothly converge toward their true value: we are in the position to "observe in action" the (vector) fixed point nature of our problem.

In our case, the convergence is slow. This is due to the fact that β is close to unity: what happens in the future matters a lot, hence the initial arbitrary values for $V(k_1)$ do not lose rapidly their weight.

The nice feature of this approach consist in the fact that it is immediate to change the interval for the state variable and the number of gridpoints, therefore adapting a numerical routine to a new situation. For example, we can use the program written for the problem above to study what happens for $k \in [0.7, 1.1]$ with 1600 gridpoints. Figure 5a plots the maximum value function, while Figure 5b shows the differences between the true value function and the approximated one. Notice that the computing time increases dramatically: it takes about seventy-five minutes to achieve convergence.

[24] In the Appendix, we provide some details about the routines used to solve our examples.

[Insert Figure 5]

This is probably the right time to became aware of one of the sad facts of life, which is usually referred to as "the curse of dimensionality". By these words, one usually refers to the fact that the speed of a procedure such as the one we have briefly described decreases more than proportionally with the number of state variables. To understand this fact, bear in mind that, to find the value function by iteration, we have discretized the state space. In our case, the state space is a line, because we have just one state variable, that is physical capital. If we had two state variables, say physical and human capital, we would have a 2-dimensional state space (which means, a plane). In this case, it would have been necessary to divide the state space is small "squares", the number of which is of course the square of the number of points we use to discretize each dimension. Accordingly, the number of "evaluation points" increases with the power of the number of state variables. Hence, the required computer time grows with the power of the number of state variables. [25] For example, in a model where the representative agent decides her consumption level and her schooling effort as functions of both human and physical capital, we need to compute the value function for every combination of human and physical capital in the two-dimensional grid.

In short: if our problem is large, we need to use a more efficient technique, which is the topic of the next Sub-section. Nevertheless, the value function iteration method can be of some help when one or more dimensions of the state space are naturally discrete. This happens when the researcher wishes to study problems where the agent may be employed or unemployed, may decide whether to get a degree or not, whether to retire or not, and so on. This method can be of some help also if the transition function is subject to a constraint. Suppose, to mention the most famous example, that investment is irreversible, so that $k_1 \geq k_0(1-\delta)$. This problem can be easily dealt with in our framework: it is sufficient not to consider the consumption level that imply an infringement of the irreversibility constraint. [26]

[25] If one chooses a different number of grid points for the two state variables–therefore dividing the plane in small rectangles–the argument in the main text must be properly adapted.

[26] The presence of an irreversibility constraint is much more interesting in a stochastic model than in a deterministic one. Figure 3 suggests that whenever the initial capital stock is below the steady-state, capital grows over time until it reaches its steady state value. Accordingly, an irreversibility constraint can bind only if our representative agents receives, as a gift, a capital level that is so much higher than the steady state one, that she wants to decumulate it at a fast rate. Instead, in a stochastic model, a shock may be sufficient to make the irreversibility constraint binding.

6.2. **Collocation techniques.** As already remarked, the previous technique is rather inefficient, because it requires the evaluation of the value function in a large number of points. Here, we present a more efficient method to solve dynamic models, which is known as "collocation technique".

Because the application of this technique to a Bellman equation is relatively complex, we move in small steps. We first underscore that the collocation technique is a way to "approximate" or to "interpolate" a function. Hence, it can be applied to any function, say to trigonometric or to exponential ones, and not necessarily to a maximum value function. Second, we show how this technique can be used to solve a simple – actually the simplest – differential equation, and finally we deal with a Bellman problem.

To understand the idea underlying this technique, we need to introduce the Weierstrass Theorem, which says:

Theorem 3. *Any continuous real function $f(x)$ can be approximated in a interval X by a polynomial $p_n(x) = \sum_{i=0}^{n} a_i x^i$.*

The words "can be approximated" mean that for any number $\nu > 0$, there is a polynomial, of suitable degree, such that $\max_{x \in X} |f(x) - p_n(x)| < \nu$. In practice, a reduction in ν, i.e. in the approximation error, usually calls for an increase in the degree of the polynomial.[27]

This is interesting and rather intuitive, but it does not tell us how to proceed in practice. Hence, let us consider a specific example, choosing a relatively challenging function: imagine that we need to approximate $\sin(x)$ over the interval $X = [0, 2\pi]$. Let us start with the third degree polynomial, $p_3(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$. Our problem is to determine the four a_i coefficients so that our polynomial behaves as the $\sin(x)$ function in the interval X. Because we need to determine four unknown coefficients, we choose four points in X. In fact, evaluating $\sin(x)$ and $p_3(x)$ at these points, we obtain four equations in the a_i's, which can be solved for the unknown coefficients.

Notice that, when the irreversibility constraint is binding, our consumer is forced to consume less than what would be optimal. Hence, she would like to avoid finding herself in this situation.
Therefore, the awareness that then irreversibility constraint may be binding, even if with a low probability, is sufficient to influence the representative agent behavior. We shall briefly return on models with stochastically binding constraints in Section (10.2).

[27] Bear in mind, however, that the relation between ν and the degree of the approximating polynomial is not always monotonic.

For example, choose $x = \{0, 2/3\pi, 4/3\pi, 2\pi\}$.[28] At these points, the collocation method forces the approximating polynomial to be identical to the function which needs to be approximated. This yields:

$$\left\{\begin{array}{c} 0 = a_0 \\ 0.86602 = a_0 + a_1 2.09440 + a_2 4.38649 + a_3 9.18704 \\ -0.86602 = a_0 + a_1 4.18879 + a_2 17.54596 + a_3 73.49636 \\ 0 = a_0 + a_1 6.28318 + a_2 39.47842 + a_3 248.05021 \end{array}\right. \quad (6.3)$$

Solving for the a_is this linear system, one obtains that the approximating third-degree polynomial is: $0 + 1.860680x - 0.888409x^2 + 0.094263x^3$.

For a "first impression judgement" of our results, we plot our polynomial and the $\sin(x)$ function over $[0, 2\pi]$. In Figure 6, the continuous line represents $\sin(x)$, while the dashed line is its third degree approximation. Some further calculations show that the largest difference between the two functions is 0.2554. This is a large number, nevertheless we can be fairly satisfied with our exercise: we have been able to approximate a trigonometric function by means of a simple third degree polynomial.

[Insert Figure 6]

However, we must be aware that our approximation deteriorates rapidly as we exit from the interval $X = [0, 2\pi]$ (compute $p_3(x)$ at 2.5π!). The rapid deterioration of the approximation outside the interval is a general characteristic of the results obtained by means of this approach. The approximation of a function by means of the collocation method is similar to what one can do with some pins and a flexible plastic stick: one takes the stick (the polynomial) and the pins (the value of the polynomial at the collocation points), and use the pins to fix the stick over the function. With a decent number of pins, one can easily make a good job in the interval inside the first and the last pin; outside this interval, the plastic stick keeps its original shape.

It is now time to leave aside once again paper and pencil. Choosing, for example, eleven collocation points, and hence using a tenth degree polynomial, we can easily build an eleven equations system. We can then use the equation matrix manipulation facilities that are built-in in computation programs such as Matlab or Gauss to solve the system; with Matlab this should take about 60/100 of a second.[29] (You can deal with this problem also by means of Mathematica or Maple). Figure 7 shows the approximation error

[28] Using the jargon, we say that we have chosen four evenly spaced "collocation points" or "nodes".

[29] A system like (6.3) can be written in matrix form as:

for this exercise. The bright side is that we are really making a decent job, the somehow dark side is that the errors are definitely higher than average at the beginning and at the end of the interval. This is usually regarded as a clear sign that we can improve our approximation.

[Insert Figure 7]

It is now possible to climb a step of the ladder that is bringing us to the solution for the dynamic optimization problem. Up to now, we have shown that we can approximate a function by means of a polynomial. Now consider that the solution for a difference or a differential equation *is* a function (in our example the solution is a function of time, which is the state variable in this exercise). This suggests that we can approximate the solution for a functional equation by means of our collocation technique.

As before, we choose a simple example, actually, we choose a differential equation that we are able to solve analytically. This will allow us to judge the "goodness of fit" characterizing our exercise.

Suppose that we need to solve the first order linear differential equation

$$\frac{dx(t)}{dt} = 0.1x(t) + 1, \text{ given } x(0) = 2, \tag{6.4}$$

in the interval $t \in [0, 4]$. Assume that the solution $x(t)$ must be approximated by a second-degree polynomial: denoting with an upperbar the approximated solution, we have

$$\bar{x}(t) = a_0 + a_1 t + a_2 t^2. \tag{6.5}$$

A good approximation, behaving "almost like" the true solution $x(t)$, must fulfill equation (6.4). Actually, the true solution must satisfy (6.4) for any t, hence, it is more than reasonable to require that the approximation fulfills (6.4) "at least somewhere", which means, in the collocation points. There, the approximation must be such that:

$$\frac{d\bar{x}(t)}{dt} = 0.1\bar{x}(t) + 1,$$

which means, exploiting (6.5) in both sides of the last equation, that at the collocation points it must be true that:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 2.09440 & 4.38649 & 9.18704 \\ 1 & 4.18879 & 17.54596 & 73.49636 \\ 1 & 6.28318 & 39.47842 & 248.05021 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.86602 \\ -0.86602 \\ 0 \end{bmatrix},$$

it is then easy to let an appropriate software solve it.

$$a_1 + 2a_2 t = 0.1(a_0 + a_1 t + a_2 t^2) + 1.$$

To determine the *three* coefficients we now "collocate" the above equation in *two* points: the third equation for our system is provided by the initial condition $\bar{x}(0) = a_0 = 2$. The two collocation points are the two extrema of the interval (0 and 4): had we chosen two different points, we would have induced large errors in some portion of the interval of interest. Accordingly, the system determining the a_i coefficients is:

$$\begin{cases} a_1 = 0.1a_0 + 1 \\ a_1 + 8a_2 = 0.1a_0 + 0.4a_1 + 1.6a_2 + 1 \\ a_0 = 2 \end{cases} . \tag{6.6}$$

A few calculations suffice to obtain the approximating function:[30] $\bar{x}(t) = 2 + 1.2t + 0.075t^2$; in Figure 8 the approximating function is the dashed line, while the continuous line portrays the exact solution (that is: $x(t) = -10 + 12e^{0.1t}$).

[Insert Figure 8]

[Insert Figure 9]

Figure 9 shows the errors that we obtain when we let our computer approximate the solution by means of an 8-degree polynomial. The continuous line represents the errors computed as differences between the exact solution and the approximated one. In this case, the eight collocation points are equally spaced between 0 and 4.

If we stop and think for just a second, we realize that the analysis of these errors is of no practical interest: why should we bother with a numerical solution when we can compute the exact one? This observation brings us close to a crucial point: when we do not know the analytic solution for a functional equation, how can we judge the goodness for the approximation?

The standard answer is grounded again on equation (6.4): because the true function satisfies this equation for any t, a good approximation should do the same. Hence, we compute the "residual function":

$$R(t) = \frac{d\bar{x}(t)}{dt} - 0.1\bar{x}(t) - 1, \;\; t \in [0, 4],$$

[30] A linear system like (6.6) can be written as:

$$\begin{bmatrix} -0.1 & 1 & 0 \\ -0.1 & 0.6 & 6.4 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix},$$

and then solved using, for example, a matrix inversion routine.

and we inspect it. If this function is "almost zero", the approximation is good.[31]

The residual function for our example is depicted by the dashed line in Figure 9.

While the approximation is very good, it can be improved: notice that the residuals are, once again, higher than average at the beginning and at the end of the interval, suggesting that some improvement is possible.

Eventually, we are ready to cope with the real problem. We now solve:

$$V(k_0) = \max_{c_0} \left\{ \ln(c_0) + 0.97 V(k_1) \right\},$$
$$s.t.\ k_1 = 3.43643 k_0^{0.3} + 0.85 k_0 - c_0,$$
$$k_0 \text{ given.}$$

for $k_0 \in [0.5\hat{k}, \hat{k}]$. Notice that this is the same problem we analyzed is Subsection 6.1, but for the fact that the depreciation parameter now takes the much more sensible value $\delta = 0.15$. Notice also that we are dealing with a quite large interval for the state variable.

Suppose first that the maximum value function is approximated by an ordinary $n - th$ degree polynomial: $V(k) = \sum_{i=0}^{n} a_i k^i$. Exploiting the first order condition, $1/c_0^* = 0.97 V'(k_1)$, one immediately obtains that:

$$c_0^* = \frac{1}{0.97 \sum\limits_{i=1}^{n} i\ a_i k_1^{(i-1)}}.$$

Substituting the approximating function and the first order condition above in the original problem, we obtain:

$$\sum_{i=0}^{n} a_i k_0^i = \left\{ -\ln(0.97) - \ln\left(\sum_{i=1}^{n} i\ a_i k_1^{(i-1)} \right) + 0.97 \sum_{i=0}^{n} a_i k_1^i \right\}.$$

Notice that the unknowns in the equation above are $n+2$: in fact, these are the $n+1$ coefficients (the a_is), and the next-period state, k_1.

If we couple the equation above with the dynamic constraint, we obtain a two equations system with $n+2$ unknown, like

$$(6.7)\quad \begin{cases} \sum_{i=0}^{n} a_i k_0^i = \left\{ -\ln(0.97) - \ln\left(\sum\limits_{i=1}^{n} i\ a_i k_1^{(i-1)} \right) + 0.97 \sum\limits_{i=0}^{n} a_i k_1^i \right\} \\ k_1 = 3.4364 k_0^{0.3} + 0.85 k_0 - \frac{1}{0.97 \sum\limits_{i=1}^{n} i\ a_i k_1^{(i-1)}} \end{cases}.$$

[31] Obviously, it is important to specify a formal criterion to evaluate the behavior of the residuals. However, at this introductory level of the analysis, we omit the discussion of this point. Nevertheless, be prepared to cope with this issue when dealing with formal research.

Suppose now to consider *two* level for k_0 (i.e. we consider two collocation points). In this case we have a system of four equations, with $n+3$ unknowns: (the $n+1$ coefficients and two next-period states). Notice that we have moved one step in the direction of the determination of the system: now the number of "missing equation" is $n+3-4=n-1$. Hence, if we choose $n+1$ collocation points (which means $n+1$ values for $k_0 \in [0.5\hat{k}, \hat{k}]$), we have $2n+2$ unknowns (the $n+1$ coefficients and the $n+1$ values for k_1, one for each collocation point), and we can build a system composed of $2n+2$ equations. In fact, for each of the $n+1$ collocation points we can build a 2-equation system like (6.7), and then we assemble a $2n+2$ system considering all the collocation points. (In other words, system (6.7) is the building block of the large system we need to solve to determine the $n+1$ a_is coefficients and the $n+1$ k_1s values.)

System (6.7), besides being large, is non linear. To deal with this problem, we can rely on the non-linear equation solver that is embedded into a software like Matlab or Gauss.[32]

However, a glance at (6.7) is enough to convince oneself that our system is *very* non linear. Hence, one might end up having troubles with the numerical routine. [33]

An often used way out is to determine the consumption function starting from the Euler equation, which, in our specific case, readily yields $c_1 =$

[32] If you do not know what a non-linear equation solver is, imagine that – having forgotten the standard formula – you need to solve a second order equation.
In this case, you may draw a second order polynomial on a plane, so that the solution(s) you are looking for are the intersection points $f(x) = 0$. To be specific, draw the polynomial upward oriented. This function, $y = f(x)$, intersects the x-axis twice, once, or never. If there are no intersection points, there are no real solutions for the equation $f(x) = 0$. Hence, draw another second order polynomial.
Now, pick a value for x and call it x_0; this is the "inital condition" for the procedure. Evaluate the function at x_0, obtaining $y_0 = f(x_0)$.
We can now sketch a naive and oversimplified numerical equation solving procedure. This is made up of a set of instructions like the following ones.

If $f(x_0) > 0$, and $f'(x_0) > 0$, then decrease x_0 by a small amount Δx.
If $f(x_0) > 0$, and $f'(x_0) < 0$, then increase x_0 by a small amount Δx.
If $f(x_0) < 0$, and $f'(x_0) > 0$, then increase x_0 by a small amount Δx.
If $f(x_0) < 0$, and $f'(x_0) < 0$, then decrease x_0 by a small amount Δx.

Notice that, following the instuctions above, one moves toward a solution for the second order polynomial. One should apply the above set of instructions over and over again, until a solution $f(x) = 0$ is reached.
Notice also that – whenever the non-linear equation admits more than a solution – the initial condition is crucial to determine which of the possible solutions is selected. Finally, notice that the initial condition is relevant to determine the time needed to converge to the solution. Miranda and Fackler [2003] and Judd [1998] provide compact treatments of techniques used in computational economics to find the solution of a system of nonlinear equations.

[33] Although it is possible to solve a system like (6.7), one often needs to feed the non-linear equation solver with a very precise initial condition, and this can be unpleasant.

$c_0\left(k_1^{-0.7}+0.8245\right)$. Assuming that $c_t=h(k_t)=\sum_{i=0}^{n}d_ik_t^i$, we have, for each collocation point, a system like:

$$\left\{\begin{array}{l}\sum\limits_{i=0}^{n}d_ik_1^i=\sum\limits_{i=0}^{n}d_ik_0^i\left(k_1^{-0.7}+0.8245\right)\\ k_1^i=3.43643k_0^{0.3}+0.85k_0-\sum\limits_{i=0}^{n}d_ik_0^i\end{array}\right., \tag{6.8}$$

The system above involves two equations and $n+2$ unknowns (the $n+1$ coefficients (the d_is) and the next-period state, k_1, which is related to the k_0 characterizing our collocation point).

It is tempting to conclude that system (6.8) is the building block of a $(2n+2)$–equations system. In other words, it is tempting to conclude that one can now choose $n+1$ collocation points, and solve a $(2n+2)$–equation system (composed of $n+1$ blocks like (6.8)) for the $n+1$ k_1^is and for the $n+1$ d_is. This is *not* the best way to carry on. In fact, in this way we omit to consider an important information, which is the knowledge that, in the steady state,

$$\hat{c}\left(=\sum_{i=0}^{n}d_i\hat{k}\right)=3.43643\hat{k}^{0.3}-0.15\hat{k}. \tag{6.9}$$

Loosely speaking, if we do not consider this information, our routine can give us a solution characterizing consumption on one of the (infinite) non optimal paths where the Euler equation is satisfied (refer back to Figure 3). To encompass the piece of information provided by the steady state, we choose n collocation points for a n^{th} degree polynomial. In this way, we have $2n+1$ unknowns ($n+1$ coefficient plus n k_1^is) and $2n$ equations (from system (6.8) we obtain two equations for each of the n collocation point), and the steady state equation above closes the system, which can actually be solved rather easily. Figure 10a and 10b show the consumption function and the residuals. These have been obtained using a tenth degree polynomial; the computation time is about 35/100 of a second.

[Insert Figure 10]

Our exercises have been based on ordinary polynomials. This choice is usually considered inefficient: ordinary polynomials have some unpleasant characteristics. As already underscored, the residuals are often concentrated in sub-intervals of the approximation interval; more importantly, it may happen that the approximation error rises, rather than falls, with the number of collocation points.

Notice also that we have always chosen evenly spaced collocation nodes. There is no reason to believe that this is the optimal choice for collocation points, nor the Weierstrass theorem provides any guidance on how to determine these points.

In practice, an often used procedure is to adopt a particular type of polynomials, the Chebychev polynomials, which are associated to non-evenly spaced nodes (the Chebychev collocation points). This procedure allows to cope with both the problems we have mentioned above. Having understood the logic of the collocation technique, it should not be a big problem to apply it by means of Chebychev nodes and polynomials, following an advanced textbook such as Judd [1998] or Miranda and Fackler [2003].

Also, having understood the collocation technique, it should not be prohibitive to deal with "finite element" methods. While the collocation technique uses as interpolating function a polynomial which can take non-zero values over the entire approximation interval, a finite element method uses interpolating functions that are *defined* to be zero in large part of the approximation interval, while they are assumed to be a (low degree) polynomial in a specific sub-interval of the approximation interval. Refer, again to Judd or to Miranda and Fackler.

The logic of the collocation methods bears some resemblance also with the one grounding the "minimum weighted residuals" technique. In fact, this method is based on the choice of some nodes and of an interpolating function, too. In this case, however, the coefficients of the polynomial are chosen in a way to minimize the (weighted) differences between the value of the function and the polynomial at the nodes. On this, see again Judd, or McGrattan [1999]

7. A Lagrangian approach to infinite-horizon stochastic growth

In our lives, very few things (if any!) can be hold as "completely sure". Accordingly, in our representative−agent growth model, the consideration of some form of risk is an important step in the direction of realism. In our framework, many things can vary stochastically over time: productivity, depreciation rate, preferences.... In this Section, we introduce an element of uncertainty by means of a simple example, that shall be solved through the Lagrangian approach; in the next section we tackle a more general framework using dynamic programming.

In the example we use to introduce the topic, we consider a Cobb-Douglas production, $A_t k_t^{\alpha}$, in which the productivity parameter A_t is stochastic. In particular, we assume that A_t can take two values, A_t^H and A_t^L, which take place with probabilities p^H and p^L $\left(\equiv 1-p^H\right)$, respectively. Of course, the superscript H stands for high, while L stands for low (hence $A_t^H > A_t^L$). We also assume that the probability for each realization is time independent, and that the two values, A_t^H and A_t^L, do not change over time. This means that the random variable A_t is identically distributed and independent over time. Although we could take as understood the subscript t, we prefer *not* to simplify the notation, in the attempt to be clearer.

In short, output is obtained by means of the stochastic production function:

$$y_t = A_t k_t^\alpha = \begin{cases} A_t^H k_t^\alpha, \text{ with probability } p_H \\ A_t^L k_t^\alpha, \text{ with probability } 1-p_H \end{cases} . \tag{7.1}$$

It is important to remark that period t productivity is supposed to be known by the representative agent when she decides upon her period t consumption.

Preferences are logarithmic, and they do not vary over time. Labeling as before the present as period 0 to save on notation, our consumer's preferences can be summarized by the following intertemporal utility function:

$$W_0 = \sum_{t=0}^{\infty} \beta^t \ln(c_t). \tag{7.2}$$

Notice that the representative agent's objective is to maximize the expected value of (7.2), i.e.

$$E_0\left[W_0\right] = E_0\left[\sum_{t=0}^{\infty} \beta^t \ln(c_t)\right],$$

where by $E_0\left[.\right]$ we denote the expectation conditional on the time 0 information set. Notice that−even in this simple case−writing in details the objective function is challenging. Its first addendum is simply $\ln(c_0)$; as for period 1 we can write: $\beta\left[p^H \ln(c_1^H) + p^L \ln(c_1^L)\right]$, where c_1^H and c_1^L are the consumption levels that will be set in place in the next period, depending on the period 1 realization for productivity. At time 2 the situation begins to become fairly complex. In fact, the period 2 consumption levels will depend not only on that period realization for productivity, but also on the productivity level that prevailed at time 1. In fact, that productivity influences period 1 consumption and, in that way, the period 2 capital level. Accordingly, we should distinguish four cases. If one follows this line of reasoning, one would get lost fairly soon: the number of cases to be taken into account grows exponentially with the time period! [34] Indeed, consider that this effort would be useless: our representative agent's problem is to choose the $c_t s$ in a "sequential" way: i.e., she chooses a c_t knowing k_t, and having observed the time t realization for productivity. Hence, in every period t, what matters is only the next-period technological uncertainty (which is limited to the random occurrence of the two states A_{t+1}^H, and A_{t+1}^L).[35]

Accordingly, we can write the representative consumer's problem by forming the present value stochastic Lagrangian in a way that underscores the "sequential" nature for our problem.

$$\begin{aligned} L_0^i \;=\; & \ln(c_0^i) - \lambda_0^i[k_1 - A_0^i k_0^\alpha + c_0^i] + \\ & + E_0\left[\beta \ln(c_1^i) - \beta\lambda_1^i(k_2 - A_1^i k_1^\alpha + c_1^i) + \right. \\ & + E_1\left[\beta^2 \ln(c_2^i) - \beta^2\lambda_2^i(k_3 - A_2^i k_2^\alpha + c_2^i) + \right. \\ & + E_2\left[\beta^3 \ln(c_3^i) - \beta^3\lambda_3^i(k_4 - A_3^i k_3^\alpha + c_3^i) + \right. \\ & + ... \;]]] \\ & + \lim_{t\to\infty} E_0\left[\beta^t \lambda_t^i k_t\right] \end{aligned} \tag{7.3}$$

Some remarks are in order. First, at time 0 the representative agent decides upon her consumption level and upon her period 1 resources (k_1), knowing the level of available resources (i.e. k_0) and the current state of productivity. Hence, the Lagrangian depends on the current productivity (one could write a Lagrangian for every productivity level). This is why we have denoted the Lagrangian with a superscript i, where $i = \{H, L\}$.

Second, at time 1, our representative agent decides upon her period 1 consumption level and upon k_2, knowing the period 1 productivity level and k_1. *At that time*, the agent maximizes:

$$\begin{aligned} & \beta \ln(c_1^i) - \beta\lambda_1^i(k_2 - A_1^i k_1^\alpha + c_1^i) + \\ & \qquad\qquad + E_1\left[\beta^2 \ln(c_2^i) - \beta^2\lambda_2^i(k_3 - A_2^i k_2^\alpha + c_2^i) + E_2\left[\ ...\ \right]\right]. \end{aligned}$$

Notice that, at time 0 the representative agent is aware of the fact that−in period 1−she will be able to choose a consumption level contingent on the future realization for productivity (and, of course, on k_1). Accordingly, what we shall do is to consider, first, the optimality conditions for period 1 *from*

[34]Bear in mind that we are considering the simplest possible case: usually, one would like to deal with random variables with more than two realization per period. Moreover, productivity is not time-independent, but highly correlated. Taking account these relevant aspects would make the problem much more intricate.

[35]In the jargon, our problem is of the "closed loop" type and not of the "open loop" kind, as it would have been if the values for the control variable(s) had been decided upon at time 0. For some further details, see Chow [1997], Chapter 2.

the perspective of that period, and then we shall consider the implications of these first order conditions from the vantage point of period 0. Of course, this line of reasoning is applied also to the subsequent periods.

Third, notice that from period 1 onward, the Lagrange multipliers become stochastic variables: they represent the marginal evaluations of consumption, that is in itself stochastic, depending on the realization for productivity.

Finally, bear in mind the last line in (7.3) is the stochastic version for the tvc (compare with Eq. (2.6)).[36]

What we do now is to consider, first, the period 0 first order conditions; we then obtain the optimality conditions for period 1 from the perspective of that period, and finally we consider the implications of the period 1 optimality conditions from the perspective of the initial period.

The variables decided upon in period 0 are: c_0^i, k_1, and λ_0^i. The first order conditions are, respectively:

$$\frac{1}{c_0^{i*}} - \lambda_0^i = 0, \tag{7.4a}$$

$$\lambda_0^i - \alpha\beta\left[p^H \lambda_1^H A_1^H + p^L \lambda_1^L A_1^L\right] k_1^{\alpha-1} = 0, \tag{7.4b}$$

$$k_1 - A_0^i k_0^\alpha + c_0^i = 0. \tag{7.4c}$$

Notice, that, in period 0, the productivity realization is known, hence the representative agent computes *one* set of first order conditions like (7.4), i.e. it computes them for $i = H$ *or* for $i = L$. The same happens when our agent optimizes at time 1 (with respect to c_1^i, k_2, and λ_1^i). However, we must consider the first order conditions for any productivity realization (i.e. for $i = \{H, L\}$). This is because we shall consider the implications of these first order conditions from the vantage point of time 0, when the realization for productivity is still unknown. The period 1 first order conditions are:

[36] In writing Problem (7.3), we have used the law of iterated expectations. This law states that $E_t\left[E_{t+n}\left[X_t\right]\right] = E_t\left[X_t\right]$, where X_t is a random variable and $n \geqslant 1$. In plain English, the law of iterated expectations tells you that what you expect today about the day after tomorrow, must be equal to what you think today that you are going to expect tomorrow about the day after tomorrow (Otherwise, you should change your current expectation!). Suppose the day you graduate, you decide to accept a job because you expect that you will earn 70.000 € in four years time. Clearly, it is not sensible that, the very day of your graduation, you believe that the following year you will expect to earn, say, 50.000 € in three years time.

$$\frac{1}{c_1^{i*}} - \lambda_1^i = 0, \tag{7.5a}$$

$$\lambda_1^i - \alpha\beta\left[p^H \lambda_2^H A_2^H + p^L \lambda_2^L A_2^L\right] k_2^{\alpha-1} = 0, \tag{7.5b}$$

$$k_2 - A^i k_1^\alpha + c_1^{i*} = 0, \tag{7.5c}$$

for $i = \{H, L\}$.

Let us now consider that, in period 0, the representative agent is aware that in period 1 she will choose her consumption knowing the realization for productivity; moreover, she is also aware that these consumption levels will be decided upon having the same information (apart from productivity) that are available at time 0. In fact, besides being contingent on productivity, the period 1 consumption levels depend on k_1, which, however is decided upon (and hence known!) at time 0. Hence, in period 0, the unique risky element that the representative agent faces in period 0 when she decides upon her consumption, is time 1 productivity. The mathematical counterpart of this reasoning emerges exploiting (7.5a) into (7.4b). Together with (7.4a) this gives:

$$\frac{1}{c_0^{i*}} = \alpha\beta\left[p^H \frac{1}{c_1^{H*}} A_1^H + p^L \frac{1}{c_1^{L*}} A_1^L\right] k_1^{\alpha-1}, \tag{7.6}$$

which can be compactly written as:

$$\frac{1}{c_0^{i*}} = \alpha\beta E_0\left[\frac{1}{c_1^*} A_1\right] k_1^{\alpha-1}.$$

This is the period 0 version for the Euler equation.

The first order conditions (7.5a-7.5c) are written from the vantage point of time 1. Of course, what our representative consumer can do at the initial time 0 is to consider that, given the available information, it must be true that:

$$E_0\left[\frac{1}{c_1^{i*}} - \lambda_1^i\right] = 0, \tag{7.7a}$$

$$E_0\left[\lambda_1^i - \alpha\beta\left[p^H \lambda_2^H A_2^H + p^L \lambda_2^L A_2^L\right] k_2^{\alpha-1}\right] = 0, \tag{7.7b}$$

$$E_0\left[k_2 - A_1^i k_1^\alpha + c_1^{i*}\right] = 0. \tag{7.7c}$$

We can now substitute the Lagrange multipliers λ_2^H and λ_2^L out of Eq. (7.7b), using the time 2 version of Eq. (7.5a). This gives the stochastic counterpart of Eq. (1.9) for period 1, which is:

$$E_0\left[\frac{1}{c_1^*}\right] = \alpha\beta E_0\left[\frac{1}{c_2^*} A_2^i k_2^{\alpha-1}\right].$$

Using this type of reasoning, one obtains that, in general, the Euler equation is:

$$E_0\left[\frac{1}{c_t^*}\right]=\alpha\beta E_0\left[\frac{1}{c_{t+1}^*}A_{t+1}^i k_{t+1}^{\alpha-1}\right], \tag{7.8}$$

which tells us that the period t expected marginal utility of consumption must be equal to the discounted expectation of the product between period $t+1$ marginal utility of consumption and productivity of capital. The representative agent considers the expectation of this product because a high marginal productivity of capital induces an increase in consumption (because of the high output level), and hence a reduction in its marginal utility, and this correlation must be taken into account. [37]

It is easy to see that it must also be true that

$$E_0\left[k_{t+1}\right]=E_0\left[A_t^i k_t^\alpha\right]-E_0\left[c_t^{i*}\right], \tag{7.9}$$

(to obtain the above equation, simply consider the proper version of Eq. (7.7c)).

We now work out the details of the explicit solution for the present example.

To determine the consumption levels, it is necessary to guess the consumption function. Our tentative solution is $c_0^{i*}=\chi A_t^i k_0^\alpha$ where χ is an undetermined parameter. Notice that Eq. (7.6) must pin down the *same* χ for any i: this is precisely due to the fact that $\chi-$ being a constant $-$ must be independent from the realization for A, and therefore from the initial state we pick.

Choosing, for example, Eq. (7.6) with $i=H$, we obtain:

$$\frac{1}{\chi A_0^H k_0^\alpha}=\alpha\beta\left[\frac{p^H A_1^H}{\chi A_1^H k_1^\alpha}+\frac{p^L A_1^L}{\chi A_1^L k_1^\alpha}\right]k_1^{\alpha-1}.$$

Hence, simplifying when possible,

$$\frac{1}{A_0^H k_0^\alpha}=\alpha\beta\left[p^H+p^L\right]k_1^{-1}.$$

When $c_0^{H*}=\chi A_0^H k_0^\alpha$, then $k_1=(1-\chi)A_0^H k_0^\alpha$, hence the above equation reduces to:

$$\frac{1}{A_0^H k_0^\alpha}=\alpha\beta\frac{1}{(1-\chi)A_0^H k_0^\alpha},$$

[37] In fact, the presence of productivity shocks causes some volatility in consumption, and in its marginal utility, and this affects welfare and decisions of a risk-averse agent.

where we exploit the fact that $p^H + p^L = 1$. The above equation is satisfied for $\chi = (1-\alpha\beta)$. Therefore, we have an explicit formulation for the consumption function, which is:

$$c_t^{i*} = (1-\alpha\beta)A_t^i k_t^\alpha. \tag{7.10}$$

The policy function above makes it explicit that the optimal consumption is chosen conditionally on the value assumed by the state variables, capital and productivity.[38]

For completeness, we now check that our solution satisfy the stochastic Euler equation (7.8).

Using Eq. (7.10) into (7.8), we obtain:

$$E_0\left[\frac{p^H}{(1-\alpha\beta)A_t^H k_t^\alpha} + \frac{p^L}{(1-\alpha\beta)A_t^L k_t^\alpha}\right] =$$

$$= \alpha\beta E_0\left[p^H \frac{1}{(1-\alpha\beta)A_{t+1}^H k_{t+1}^\alpha} A_{t+1}^H k_{t+1}^{\alpha-1} + p^L \frac{1}{(1-\alpha\beta)A_{t+1}^L k_{t+1}^\alpha} A_{t+1}^L k_{t+1}^{\alpha-1}\right].$$

Simplifying where possible, we obtain:

$$E_0\left[\frac{p^H}{A_t^H k_t^\alpha} + \frac{p^L}{A_t^L k_t^\alpha}\right] = \alpha\beta E_0\left[\frac{p^H}{k_{t+1}} + \frac{p^L}{k_{t+1}}\right].$$

Because, in any state i, $k_{t+1} = A_t^i k_t^\alpha - c_t^{i*}$, and $c_t^{i*} = (1-\alpha\beta)A_t^i k_t^\alpha$, then $k_{t+1} = \alpha\beta A_t^i k_t^\alpha$. Hence, the above equation becomes:

$$E_0\left[\frac{p^H}{A_t^H k_t^\alpha} + \frac{p^L}{A_t^L k_t^\alpha}\right] = \alpha\beta E_0\left[\frac{p^H}{\alpha\beta A_t^H k_t^\alpha} + \frac{p^L}{\alpha\beta A_t^L k_t^\alpha}\right],$$

which is obviously verified.

Exercise 9. *Modify the example in Section 7, assuming that*

$$W_0 = \sum_{t=0}^{3} \beta^t \ln(c_t),$$

and find the consumption function.

Exercise 10. *Modify the example in Section 7, assuming that the time $t+1$ realization for productivity depends upon its time t realization. In particular, assume that $\Pr(A_{t+1}^H \,|A_t^H) = \Pr(A_{t+1}^L \,|A_t^L) = p$, and that $\Pr(A_{t+1}^H \,|A_t^L) = \Pr(A_{t+1}^L \,|A_t^H) = 1-p$. Find the consumption function.*

Exercise 11. *In the example in Section 7, use the guess $c_0^i = \Psi(A_t^i)k_0^\alpha$ (where the functional relation $\Psi(A_t^i)$ between consumption and productivity*

[38] You can now verify that, choosing $i = L$, Eq. (7.6) gives the same value for χ.

is assumed to be unknown) and find the consumption function (i.e. show that $\Psi(A_t^i)$ must be linear).

Exercise 12. *Assume that: i) the single period utility function is: $c_t^{1-\gamma}/(1-\gamma)$, where $\gamma \in [0,1) \cup (1,\infty)$; ii) the dynamic constraint is $k_{t+1} = A_t^i k_t - c_t$, in which the structure for the productivity shocks is the one described in Section 7. Find the consumption function.*

8. The Bellman formulation for the stochastic problem

We now consider the dynamic programming approach to a simple stochastic version of the growth model. As before, we shall consider stochastic the productivity parameter, but our reasoning applies to the introduction of any form of uncertainty that preserves the additive separability of the objective function.[39]

Assuming that productivity is stochastic, we can formulate the consumer's intertemporal problem as:

$$\begin{gathered} V(k_0, A_0) = \max_{c_0} \left\{ U(c_0) + \beta E_0 \left[V(k_1, A_1) \right] \right\}, \\ s.t.\ k_1 = A_0 f(k_0) + (1-\delta)k_0 - c_0, \\ A_0,\ k_0 \text{ given.} \end{gathered} \tag{8.1}$$

Notice that the maximum value function in (8.1) has both capital and random productivity as its argument. Before discussing how to deal with the above problem, we should qualify the stochastic process characterizing productivity, but we postpone this discussion for a while. If you wish, for now you can imagine that the productivity process is independent over time (i.e. A_{t+1} is independent from A_t for any t).

The Bellman equation in (8.1) tells us that, to obtain the period 0 value function $V(k_0, A_0)$, it is necessary to maximize, with respect to c_0, the expression $U(c_0) + \beta E_0 [V(k_1, A_1)]$. Accordingly, a necessary condition for a maximum is obtained differentiating $U(c_0) + \beta E_0 [V(k_1, A_1)]$ with respect to current consumption, which gives:

$$U'(c_0^*) + \beta E_0 \left[V'(k_1, A_1) \right] \frac{\partial k_1}{\partial c_0} = 0, \tag{8.2}$$

[39]For example, one can deal with stochastic preferences of the type: $W_0 = \sum_{t=0}^{\infty} \beta^t U(c_t, \epsilon_t)$, where ϵ_t is a sequence of random variables. In fact, we may write $W_0 = U(c_0, \epsilon_0) + \beta \sum_{t=1}^{\infty} \beta^t U(c_t, \epsilon_t)$, which is the basis for the Bellman's formulation.

where, $\partial k_1/\partial c_0 = -1$, from the capital accumulation equation. We can exploit the transition equation in (8.1), the Bellman equation and the first order condition (8.2) to form the system:[40]

$$\begin{cases} V(k_0, A_0) = U(c_0^*) + \beta E_0 \left[V(k_1, A_1) \right], \\ k_1 = f(k_0) + (1-\delta)k_0 - c_0^*, \\ U'(c_0^*) = \beta E_0 \left[V'(k_1, A_1) \right], \\ A_0, \ k_0 \ \textit{given}. \end{cases}$$

Our task is to find the solution for the above problem. As in the deterministic case there are two ways to proceed: we can (try to) solve the problem "guessing" the solution, or we can use numerical techniques.

In finding a solution we can exploit, whenever this is useful, the envelope theorem, which in this context gives:

$$V'(k_0, A_0) = \beta E_0 \left[V'(k_1, A_1) \right] \frac{\partial k_1}{\partial k_0}.$$

The proof for this result is trivial, and it is left to the reader, who can also work out how to apply the Benveniste and Scheinkman formula.

Before applying the techniques and the results mentioned above to the solution of problem (8.1), we should specify under which conditions the solution for our problem exists and is unique. In stochastic settings, this turns out to be very complex. In practice, what people often do (in addition to verify that the conditions spelled out in Theorem 1 or in Theorem 2 are fulfilled) is to check that the stochastic process playing a role in their model enjoys the "Markov property".[41]

Here, we remind what a Markov process is, and then we explain why is important to restrict our attention to dynamic optimization problems in which the stochastic disturbances belong to this class.

Definition 1. *(Markov process). A stochastic process x_t is Markov if for every $\bar{x}$ and for every period* $0, 1, 2, ..., t, ...$ *we have*

$$\Pr \left\{ x_{t+1} \leq \bar{x} | x_t, x_{t-1} ... x_0 \right\} = \Pr \left\{ x_{t+1} \leq \bar{x} | x_t \right\}.$$

In words, a Markov process is a random process whose future probabilities are determined by its most recent realization. In fact, the above definition tells us that the probability that the next period value for the process is below a given threshold (i.e. that $x_{t+1} \leq \bar{x}$), depends only on the current realization of the process, x_t. Hence, the past realizations (x_{t-1}, ..., x_0) are actually irrelevant for the determination of $\Pr \left\{ x_{t+1} \leq \bar{x} \right\}$. Sometimes it is

[40]Alternative formulation exploits the transition equation and/or the first order condition to substitute out c_0^* and/ or k_1.

[41]For a much more detailed introduction to the issue we refer again to Thompson [2004]. Lucas, Stokey and Prescott [1989] is the key reading.

said that, with Markov processes, "history does not matter" because the current value x_t is all what is needed to compute future probabilities, and it does not matter how variable x got there. Alternatively, we can think to a Markov process as a sequence of random variables for which "memory does not matter": what we need to know is just x_t, the state of the process, and we do not need to recall the past realizations.

The fact that the random disturbances belong to Markov processes represents an enormous simplification for a dynamic optimization problem. In fact, the maximum value and the policy function can be expressed as functions of the most recent realization for the random variables. In other words, the most recent known realization for the random variables represent the *unique* additional state of the system. The random variables considered in the examples in the previous Section and in the exercises obviously belong to Markov process.[42]

9. Guess and verify, in two dimensions

In this Section, we analyze a version of the Brock and Mirman's (1972) optimal growth model, that is less trivial than the one considered in Section 7, and we show how to obtain its closed form solution.

The consumer's side of the problem is unchanged, hence her preferences are given by (4.1); the production function is Cobb-Douglas, characterized by a depreciation parameter as high as unity, while the stochastic productivity parameter A_t evolves according to:

$$\ln A_{t+1} = \rho \ln A_t + \epsilon_{t+1}, \tag{9.1}$$

where $\rho \in [0, 1]$ is the "auto-regressive parameter", and ϵ_{t+1} is a random variable that is time independent and identically distributed over time. This means that ϵ_{t+1} is not influenced by ϵ_t, ϵ_{t-1}... (hence $\ln A_t$ is a Markov process), and that the characteristics of ϵ_{t+1} do not change over time. If you wish, you can conceive ϵ_{t+1} as a normal random variable with mean 0 and variance σ_ϵ^2. Notice that ϵ_{t+1} represents the innovation in the stochastic process (9.1).

While some auto-regression in productivity is highly realistic, the structure for A_t postulated in (9.1) may seem rather *ad hoc*: it requires that it is the logarithm of productivity that depends on its past realization. Notice, however, that the particular structure postulated in (9.1) grants that productivity never becomes negative for any realization of the innovation, which

[42]When the random variables in a Markov process can take only a finite number of values, then this process is called a Markov chain. When the random variables in a Markov process are continuous, then the process is known as a Markov sequence. A dynamic optimization problem in which the stochastic variables belong to Markov processes and the payoff enter additively in known as a Markov decision process.

makes sense. Moreover, the structure in (9.1) is an essential ingredient to obtain a closed form solution, and hence for now we must live with it.

As before, the period t productivity is supposed to be know by the representative agent when she takes her period t decisions.

Our problem is now to solve:

$$\begin{aligned} V(k_0, A_0) &= \max_{c_0} \{\ln(c_0) + \beta E_0 [V(k_1, A_1)]\}, \\ s.t.\ k_1 &= A_0 k_0^\alpha - c_0, \\ \ln A_1 &= \rho \ln A_0 + \epsilon_1, \\ A_0,\ & k_0 \text{ given.} \end{aligned}$$

We now introduce our tentative solution, which takes the form:

$$V(k_t, A_t) = F + G\ln(k_t) + H\ln(A_t). \tag{9.2}$$

It is worth emphasizing that because the value function depends on two state variables, it is necessary to specify a solution involving the two states. This simple but important fact remains true also when we shall deal with numerical solutions.

The first order condition is $1/c_0^* = \beta E_0 [V_k(k_1, A_1)]$, which rapidly gives: $k_1 = \beta G c_0^*$. This expression can be used to obtain c_0^* from the dynamic constraint, which gives

$$c_0^* = \frac{1}{1+\beta G} A_0 k_0^\alpha. \tag{9.3}$$

This equation implies that the next period capital stock is:

$$k_1 = \frac{\beta G}{1+\beta G} A_0 k_0^\alpha.$$

The above equation, Eq. (9.3), and the tentative solution (9.2) must be substituted back into the Bellman equation, which gives:

$$\begin{aligned} & F + G\ln(k_0) + H\ln(A_0) = \\ & \quad = \ln\left(\frac{1}{1+\beta G} A_0 k_0^\alpha\right) + \beta E_0 \left[F + G\ln\left(\frac{\beta G}{1+\beta G} A_0 k_0^\alpha\right) + H\ln(A_1)\right]. \end{aligned}$$

Exploiting the usual properties of logarithmic functions, we obtain:

$$F + G\ln(k_0) + H\ln(A_0) =$$
$$= \ln\left(\frac{1}{1+\beta G}\right) + \ln(A_0 k_0^{\alpha})$$
$$+ \beta F + \beta G \ln\left(\frac{\beta G}{1+\beta G}\right) + \beta G\ln(A_0 k_0^{\alpha}) + \beta H E_0\left[\ln(A_1)\right],$$

or:

$$F + G\ln(k_0) + H\ln(A_0) =$$
$$= -\ln(1+\beta G) + \ln A_0 + \alpha \ln k_0$$
$$+\beta F + \beta G\ln\beta G - \beta G\ln(1+\beta G) + \beta G\ln A_0 + \alpha\beta G\ln k_0 + \beta H E_0\left[\ln(A_1)\right].$$

Notice that we have exploited the assumption according to which the period 0 realization for A_t is known at the time of choosing period 0 consumption. We now exploit (9.1) to substitute, in the equation above, $\beta H\rho\ln(A_0)$ to $\beta H E_0\left[\ln(A_1)\right]$.

Having substituted out $E_0\left[\ln(A_1)\right]$, we recall that the resulting equation must be satisfied for *any* k_0, for *any* A_0, and for *any* admissible value of the parameters β, and α. Hence, it must be that:

$$\begin{cases} G = \alpha + \alpha\beta G \\ H = 1 + \beta G + \beta H\rho \\ F = -\ln(1+\beta G) + \beta F + \beta G\ln\beta G - \beta G\ln(1+\beta G) \end{cases}.$$

From the first equation in the system above we obtain:

$$G = \frac{\alpha}{1-\alpha\beta}.$$

Exploiting this result in the second equation of the system, we get:

$$H = \frac{1}{(1-\rho\beta)(1-\alpha\beta)}.$$

Notice that $G, H > 0$, because $\alpha, \beta \in (0,1)$ and $\rho \in [0,1]$. This implies that $V_k(k_t, A_t)$, $V_A(k_t, A_t) > 0$: a richer consumer enjoys an higher overall utility, so does a consumer who lives in a more productive economic environment. Notice also that the higher ρ, the larger is $V_A(k_t, A_t)$: when productivity is more persistent, its increase has a stronger impact on the value function simply because it last longer.

Substituting G in the third equation gives:

$$F = \frac{1}{1-\beta}\left[\frac{\alpha\beta}{1-\alpha\beta}\ln(\alpha\beta) + \ln(1-\alpha\beta)\right].$$

Because F, G, and H are independent from capital and productivity, our guess (9.2) is verified, and the maximum value function is:

$$V(k_t, A_t) = \frac{1}{1-\beta}\left[\frac{\alpha\beta}{1-\alpha\beta}\ln(\alpha\beta) + \ln(1-\alpha\beta)\right] + \\ + \frac{\alpha}{1-\alpha\beta}\ln(k_t) + \frac{1}{(1-\rho\beta)(1-\alpha\beta)}\ln(A_t).$$

The value function above is strikingly similar to the one we obtained in Sub-section (4.1): this is an effect of the functional forms we have chosen to describe preferences, production, and the evolution for productivity.

It is now easy to obtain the consumption function. Exploiting G in equation (9.3), we obtain $k_1 = \alpha\beta A_0 k_0^\alpha$, hence $c_0^* = (1-\alpha\beta)A_0 k_0^\alpha$, as in the non-stochastic case. This is a neat but slightly disappointing result, because optimal consumption turns out to be, again, a linear function of output.

Exercise 13. *Find the value function, and the consumption function for the example in Section 7.*

Exercise 14. *Assume that: i) the single period utility function is:* $c_t^{1-\gamma}/(1-\gamma)$, *where* $\gamma \in [0,1) \cup (1,\infty)$; *ii) the dynamic constraint is* $k_{t+1} = A_t^i k_t - c_t$, *in which the structure of the productivity is the one described in Section 7. Find the value function, and the consumption function.*

Exercise 15. *Assume that i) the single period utility function is:* $\ln(c_t) + \gamma\ln(1-l_t)$, *where* l_t *is labour time and* $\gamma > 0$; *ii) the dynamic constraint is* $k_{t+1} = A_t k_t^\alpha l_t^{(1-\alpha)} - c_t$, *where* $0 < \alpha < 1$, *and* A_t *is described by equation (9.1). Find the value function, and the related policy functions.*

Exercise 16. *(Cake eating with taste shocks). Assume that i) the single period utility function is:* $\ln(z_t c_t)$, *where* z_t *is a random variable that can assume two values,* z^H *and* z^L *with probabilities* p^H *and* $1-p^H$; *ii) the dynamic constraint is* $y_{t+1} = y_t - c_t$.[43] *Find the value function, and the consumption function.*

Exercise 17. *Consider a Central Bank which can control output (e.g. by means of the interest rate) and aims to minimize:*

$$W_0 = E_0\left[\sum_{t=0}^{\infty}\beta^t\left(y_t^2 + \delta\pi_t^2\right)\right],$$

where π_t *is inflation and* y_t *is output. The dynamic constraint the Central Bank faces is a Phillips' curve:* $\pi_{t+1} = \pi_t + \gamma y_t + \epsilon_t$, *where* ϵ_t *is a random*

[43]Interpret y_t as the remaining share of a cake, and c_t as the slice of the cake the consumer decides to cut out at time t. The implicit assumption is that the consumer has free access to a very efficient fridge.

variable independent and identically distributed over time. Find the value function and the policy function.

10. Numerical techniques and the "curse of dimensionality"

In this Section, we illustrate two alternative techniques that can be used to approximate numerically the maximum value function in stochastic settings.

First, we extend to the case of stochastic productivity the value function iteration method that we introduced in Sub-section 6.1. We then remark that this technique bitterly suffers from the "curse of dimensionality". Finally, because the alternative we proposed for the deterministic case, namely the use of collocation techniques, can be troublesome, we discuss a simple application of a more efficient approach, which is the one based on "parameterized expectations".

10.1. **Discretization of the state space.** In this Sub-section, we present the stochastic version of the value function iteration method. To favour comparisons, we modify the example presented in Sub-section 6.1, encompassing the simplest possible stochastic process for productivity. In fact, we assume that A_t is independent and identically distributed over time, and that it can take two values, $A_t^L = 3.36770$ and $A_t^H = 3.50515$, both with probability 0.5 (so that $E_0[A_t]$ is equal to the productivity value used in the non stochastic example: what we are considering here is a mean preserving spread in productivity; notice also that $-$ being A_t time independent $-$ we drop the time suffix). As before, α is 0.3, and β is 0.97, so that the non-stochastic steady state for capital is normalized to unity.

Accordingly, our problem is to solve:

$$\begin{gathered} V(k_0, A) = \max_{c_0} \left\{ \ln(c_0) + 0.97 E_0 \left[V(k_1, A) \right] \right\}, \\ s.t.\ k_1 = A k_0^{0.3} - c_0, \\ A \in \{3.36770,\ 3.50515\}, \\ k_0 \text{ given.} \end{gathered} \tag{10.1}$$

As before, we assume to be interested in solving the problem for capital values that are around the non stochastic steady state, and we consider only five possible levels for capital: $\{0.98,\ 0.99,\ 1,\ 1.01,\ 1.02\}$; accordingly, our state space is composed of ten points: the five capital levels must be coupled with the two possible productivities.

The strategy to obtain the maximum value function is not relevantly different from the one we have described for the deterministic case.

To obtain the unknown value function, we need to:

- (Step 1) specify $E_0[V(k_1, A)]$ as a set of five arbitrary values, one for each capital level, and denote this set of initial values as $E_0[V_0(k_1, A)]$;
- (Step 2) solve Problem (10.1) for each state k_1, A, finding a new (and hence different) maximum value function (in our case this step substantiates in the attainment of a set composed of ten values, denoted by $V_1(k_0, A)$);
- (Step 3) compute the *expected* maximum value function $E_0[V_1(k_1, A)]$ using $V_1(k_1, A) = V_1(k_0, A)$, and the probability distribution over A;
- (Step 4) using $E_0[V_1(k_1, A)]$ as a new arbitrary starting point, obtain $E_0[V_2(k_1, A)]$;
- (Step 5) iterate the steps above until $E_0[V_n(k_1, A)]$ and $E_0[V_{n+1}(k_1, A)]$ are "sufficiently close".

This iterative procedure stops when the set of values representing $E_0[V(k_1, A)]$ meets the convergence criterion, which, in the simulation, will be analogous to the one chosen in the non-stochastic case.

We now illustrate the above procedure by means of an extremely simplified example.

We start by choosing the set of values for $E_0[V_0(k_1, A)]$, which is: $E_0[V_0(k_1, A)] = 0$ for any k_1 (Step 1).

As for Step 2, notice that the above choice implies that we face problems of the following type:

$$\begin{gathered} V_1(k_0, A) = \max_{c_0} \{\ln(c_0)\}, \\ s.t.\ k_1 = A k_0^{0.3} - c_0, \\ A \in \{3.36770,\ 3.50515\}, \\ k_0 \text{ given.} \end{gathered}$$

We first consider the case $A = A^L$. Notice that, as before, both k_0 and k_1 must take one of the possible values for capital (i.e. {0.98, 0.99, 1, 1.01, 1.02}): this restricts the set of feasible consumption levels. For example, if $k_0 = 0.98$, for $k_1 = \{0.98, 0.99, 1, 1.01, 1.02\}$, we compute: $c_0 = \{2.36735, 2.35735, 2.34735, 2.33735, 2.32735\}$; the corresponding utilities are {0.86177, 0.85754, 0.85329, 0.84902, 0.84473}. The highest utility level is obtained for $c_0 = 2.36735$, which is the consumption level that guarantees that the next period capital is $k_1 = 0.98$.

Hence, the solution for this specific problem is $V_1(k_0 = 0.98, A = A^L) = 0.86177$.

Repeating this reasoning we obtain: $V_1(k_0 = 0.99, A = A^L) = 0.86607$, $V_1(k_0 = 1, A = A^L) = 0.87033$, $V_1(k_0 = 1.01, A = A^L) = 0.87454$, $V_1(k_0 =$

$1.02, A = A^L) = 0.87870$. As expected, in every problem the solution corresponds to the maximum feasible consumption level, which is the one corresponding to $k_1 = 0.98$.

We then consider the case $A = A^H$. For example, if $k_0 = 0.98$, for $k_1 = \{0.98, 0.99, 1, 1.01, 1.02\}$, we compute: $c_0 = \{2.50397, 2.49397, 2.48397, 2.47397, 2.46397\}$; the corresponding utilities are $\{0.91788, 0.91388, 0.90986, 0.90583, 0.90178\}$. The highest utility level is reached for $c_0 = 2.50397$, which is, once again, the consumption level that guarantees that the next period capital is $k_1 = 0.98$, and the solution for the problem is $V_1(k_0 = 0.98, A = A^H) = 0.91788$.

As for the other capital levels, we obtain: $V_1(k_0 = 0.99, A = A^H) = 0.92211$, $V_1(k_0 = 1, A = A^H) = 0.92630$, $V_1(k_0 = 1.01, A = A^H) = 0.93044$, $V_1(k_0 = 1.02, A = A^H) = 0.93454$. This completes Step 2 in the procedure.

The third step in the procedure is readily executed: given that $p^H = p^L = 0.5$, $E_0[V_1(k_1, A)] = \{0.88982,\ 0.89409, 0.89831, 0, 90249, 0.90662\}$.

We now use this set of values as a new starting point for the maximum value function (Step 4). Accordingly, we now assume $E_0[V_1(k_1, A)] = \{0.88982,\ 0.89409, 0.89831, 0, 90249, 0.90662\}$, and we proceed with the second iteration, as we did in Sub-section 6.1. Actually, we let our numerical routine to perform Steps 4 and 5.

The results of the routine that solves the model are reported in Table 2; the last line provides, as a reference point, the exact solution.

[Insert Table 2]

Notice that the uncertainty about future productivity has an interesting impact on the value function: an increase in the variance for A negatively influences the overall utility. (Compare the last line in Table 2 with the last one in Table 1) This is a consequence of the fact that the preferences (4.1) describe a risk-averse representative consumer.

Exercise 18. *Check that the values for $V(k)$ in the last line of Table 2 are correct.*

To provide a more challenging example, we use the program written for the problem above to study what happens for $k \in [0.5, 1.3]$ with 800 gridpoints and five equiprobable productivity levels (evenly spaced in the interval $A = [3.36770,\ 3.50515]$.[44] Figure 11a plots the expected maximum

[44]When dealing with continuous random variables, one needs to use "quadrature" techniques. In practice, this amounts to a wise choice of the points used to discretize the continuous random variable. Refer to Judd [1998].

value function, while Figure 11b shows the differences between the true expected value function and the approximated one; it takes about one hour and forty-five minutes to achieve convergence.

[Insert Figure 11]

Even in this simple example, the needed computer time is high. Hence, one should consider to adopt a more efficient technique, such as the collocation one or the finite elements method. However, the "curse of dimensionality" never sleeps, and the implementation of these techniques in a multidimensional setting is not easy. In fact, the interpolating polynomial must be specified in a number of variables equal to the dimension of the state space, and the number of coefficients to be determined grows very quickly with the number of dimension.[45] Hence, the number of equation composing the non-linear system of ordinary equations that comes from the collocation exercise can easily become very large, and therefore hardly manageable for your non-linear equation solver. In practice, when the number of state variables exceeds two, the application of these techniques becomes a fairly tough task. The next section sketches a valid alternative.

10.2. **The Parameterized Expectations Approach.** As usual, we introduce this new approach by means of an example. We consider the version of the stochastic Euler equation (7.8), in which the capital does not depreciate entirely within one period, and hence $\delta \in (0,1)$. From the perspective of period t, the expectation are conditioned upon the time t information, and hence the Euler equation is:

$$\frac{1}{c_t^*} = \beta \left\{ \alpha E_t \left[\frac{1}{c_{t+1}^*} (A_{t+1} k_{t+1}^{\alpha-1} + (1-\delta)) \right] \right\}. \tag{10.2}$$

The term on the right hand side in the curly bracket is an expectation, which is conditional on the period t information set.

Notice that the term $\left[\frac{1}{c_{t+1}^*}(A_{t+1}k_{t+1}^{\alpha-1} + (1-\delta))\right]$ is a function of A_{t+1} and k_{t+1}. In fact, c_{t+1} is a function of the period $t+1$ realizations for the state variables. Hence $\left\{\alpha E_t \left[\frac{1}{c_{t+1}^*}(A_{t+1}k_{t+1}^{\alpha-1} + (1-\delta))\right]\right\}$ is the expectation of a function of A_{t+1} and k_{t+1}. These variables must be forecasted on the ground of the period t information set, which means on the ground of A_t and k_t.

[45]Remind that $P_3(x)$ was characterized by four coefficients only, and consider that an order three polynomial in two variables is characterized by ten coefficients. In fact:

$$P_3(x,y) = a_0 + a_1x + a_2y + a_3x^2 + a_4xy + a_5y^2 + a_6x^3 + a_7x^2y + a_8xy^2 + a_9y^3.$$

Hence, we conclude that these are the variables upon which the expectation on the right hand side of (10.2) must be conditioned.[46]

Because the expectation $\left\{\alpha E_t\left[\frac{1}{c^*_{t+1}}(A_{t+1}k_{t+1}^{\alpha-1}+(1-\delta))\right]\right\}$ is a function of the state variables, we approximate this conditional expectations by a polynomial in the state variables. Such a polynomial – as any polynomial – is characterized by its parameters, therefore – once we have chosen the degree of the polynomial and the values for the parameters – we have "parameterized" the expectation.

We denote the approximating polynomial by $F(\Psi_0, A_t, k_t)$, where Ψ_0 is the set of coefficients of the polynomial. Bear in mind that $F(\Psi_0, A_t, k_t)$ represents the period t conditional expectation for

$$\left\{\alpha E_t\left[\frac{1}{c^*_{t+1}}(A_{t+1}k_{t+1}^{\alpha-1}+(1-\delta))\right]\right\}.$$

Following Marcet and Lorenzoni (1999), we choose an exponentiated polynomial of order one, that is:

$$F(\Psi_0, A_t, k_t) = \psi_0^1 \exp(\psi_0^2 \ln(K_t) + \psi_0^3 \ln(A_t)). \tag{10.3}$$

Hence, in this example Ψ_0 is composed of ψ_0^1, ψ_0^2, and ψ_0^3.

Using a regular polynomial might cause problems because it can generate a negative value for $F(\Psi_0, A_t, k_t)$; since, as we shall see in a while, this number is raised to a negative power (refer to Eq. (10.4a)), a numerical error would ensue. Furthermore, we know that the true expectation can take only positive values, and the functional form in (10.3) actually guarantees a positive $F(\Psi_0, A_t, k_t)$ (and so it generates a positive solution for consumption). Increasing the degree of the exponentiated polynomial, we can approximate the conditional expectation with better and better accuracy.

We initialize our procedure attributing to each parameter (i.e. to ψ_0^1, ψ_0^2, and ψ_0^3) an arbitrary value; having chosen these values, knowing the current values for the state variables, A_0, and k_0, and letting our software to draw an appropriate sequence of random numbers representing the future realizations of the productivity process, we can easily simulate the model. As it will be commented upon in what follows, what we do is to obtain "artificial" time series for consumption and next-period capital. In fact, in any period $t = \{0, 1, 2, ...\}$, we have:

[46] You may be thinking that k_{t+1} belongs to period t information set. This is true, but consider that k_{t+1} depends on c_t. Because this is the variable we wish to determine, it is unconvenient to condition the expectation on the right hand side of (10.2) on k_{t+1}.

$$c_t(\Psi_0) = [\beta F(\Psi_0, A_t, k_t)]^{-1} \tag{10.4a}$$

$$k_{t+1}(\Psi_0) = A_t k_t^{\alpha} + (1-\delta)k_t - c_t(\Psi_0). \tag{10.4b}$$

In words, knowing A_0 and k_0, we compute $c_0(\Psi_0)$, this value is used, together with A_0 and k_0, to determine $k_1(\Psi_0)$.Having obtained a random realization for A_1, we couple this with $k_1(\Psi_0)$, and we iterate the process. Because the values for consumption and for the capital stock obtained by means of (10.4a) and of (10.4b) depend on the vector of parameters, we have denoted these values as $c_t(\Psi_0)$, and $k_{t+1}(\Psi_0)$, respectively.

Notice that this simulation is based on an *arbitrary* choice for the vector of parameters. Nonetheless, we take it seriously and we construct a time series for the auxiliary variable w_t:

$$w_t = \left\{\alpha \frac{1}{c_{t+1}(\Psi_0)}(A_{t+1}k_{t+1}(\Psi_0)^{\alpha-1} + (1-\delta))\right\}, \quad t = \{0, 1, 2, ...\} \tag{10.5}$$

Notice that

$$E_t\left[w_t\right] = E_t\left[\alpha \frac{1}{c_{t+1}(\Psi_0)}(A_{t+1}k_{t+1}(\Psi_0)^{\alpha-1} + (1-\delta))\right].$$

Hence, $E_t\left[w_t\right]$ is the marginal utility of period t consumption multiplied by $1/\beta$, as determined, following the Euler equation, on the ground of our artificial time series. In our model, $(\beta c_t^*)^{-1}$ is a function of the state variables K_t and A_t. Hence, $E_t\left[w_t\right]$ can be expressed as a function of the state variables; if the parameter vector Ψ_0 and the functional form in (10.3) were correct (i.e. were the ones actually satisfying (10.2)), by regressing – according to the functional form (10.3) – w_t on K_t and A_t (plus a constant) we should obtain exactly Ψ_0.

Because $F(\Psi_0, A_t, k_t)$ is an exponentiated polynomial, we run the regression

$$\log(w_t) = \log(\psi_1^1) + \psi_1^2 \ln(K_t) + \psi_1^3 \ln(A_t) + \xi_t,$$

where is ξ_t a shock, which – under rational expectations – must be independent over time and from the regressors (otherwise the correlations could be exploited to improve the forecast).

We denote the set of regression coefficients as Ψ_1. *If* the parameter vector Ψ_0 were correct, then the regression parameters Ψ_1 would confirm Ψ_0 (and hence the polynomial built using Ψ_1 would be equal to the original one, built on Ψ_0).

A key point is that the regression coefficients Ψ_1 can be used to simulate again the model. In practice, we substitute the vector Ψ_1 to the initial arbitrary vector Ψ_0, and then we obtain new values for $c_t(\Psi_1)$, $k_t(\Psi_1)$, and for the related time series w_t. If we proceed in this way, we can obtain a new estimate Ψ_2. The crucial aspect is that iterating this reasoning, we obtain better and better result, i.e. we obtain time series for w_t which are better and better approximations for $(\beta c_t^*)^{-1}$. (See Marcet and Lorenzoni (1999) and the literature quoted there for more details on this convergence result).

We now solve by means of this procedure the log-utility/Cobb Douglas production model.

Needing to specify the parameters' values, we fix, as usual, $\beta = 0.97$, and $\alpha = 0.3$; in coherence with the example in Section 6.2, we pick, for the depreciation parameter, the value $\delta = 0.15$; for the productivity process we choose, as in the previous Sub-section, $A_t^L = 3.36770$, and $A_t^H = 3.50515$, both with probability 0.5. The initial condition for the capital stock is $k_0 = 6$. We choose an initial condition quite far from the long-run capital distribution to obtain some information about the "transition" of the system to the stochastic steady state.

Finally, we need to provide the initial values for the parameters in Ψ_0. Although we have defined as "arbitrary" these values, it is sensible to feed the routine with the values that are closest to the truth: this speeds up the convergence. [47] Accordingly, we have chosen for ψ_0^i, the values that can be computed when $\delta = 1$ (solve Exercise 13). These are: $\psi_0^1 = 1.4540590$, $\psi_0^2 = -0.3$, and $\psi_0^3 = -1$. [48]

We choose to simulate the model for 100.000 periods, and we assume that convergence is attained when the largest difference in the computed values for a single parameter between an iteration and the successive one, in absolute value, is lower than $10^{(-9)}$. It turns out that convergence is reached in 44 iteration, the required computer time being about 3' and 40". [49]

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[47] Actually, values that are far from the correct ones may prevent the routine to converge. Also, it may be interesting to underscore that several applications of PEA use an algorithm based on successive approximations. Denoting by $\tilde{\Psi}$ the parameter vector estimated in the least squares section of the procedure, one picks $\Psi_n = \mu\tilde{\Psi} + (1-\mu)\Psi_{n-1}$, with $\mu \in (0, 1)$, instead of choosing $\Psi_n = \tilde{\Psi}$.

[48] The difference between $\delta = 1$ and $\delta = 0.15$ is large. Thus, our starting values for the parameters can actually be far from the "true" ones. Accordingly, the initial values may induce instability. In this case, it is sensible to carry on as follows. First, run the routine for a δ close to 1, say $\delta = 0.9$, obtaining the "true" values for this case. Second, reduce δ, say to 0.8, using as a starting point for Ψ, the "true" values obtained in the first step. This should allow for the computation of a new set of "correct" values for Ψ. One can progress this way until the desired value ($\delta = 0.15$) is obtained.

[49] As already underscored, when the parameter vector Ψ_n is correct, then $E_t[w_t] = (\beta c_t^*)^{-1}$. Accordingly, we computed the average difference between w_t and $(\beta c_t^*)^{-1}$, and we found it to be very small: -5.10$x10^{(-7)}$.

Figure 12a shows the scatter plot for consumption as a function of capital; Figure 12b provides an idea of the evolution for the parameters values: the continuous line represents ψ_j^1; the dotted line is ψ_j^2, while the dashed line is ψ_j^3.

[Insert Figure 12]

Figure 12 shows that k_t takes only a relatively limited number of periods to move from its initial value ($k_0 = 6$) to its steady state distribution. Accordingly, the transition is affected only by a limited number of realization for the productivity shock. Hence, the time series we obtain for the endogenous variables cannot be taken as describing a "typical" behavior.

The bright side of what we have done, is that we have been able to compute a good approximation in a very short time. In general, the Parameterized Expectations Approach does well in solving dynamic models: for example, Christiano and Fisher [2000] argue that it should be preferred to solve models with stochastically binding constraint.

In sum, this is a method that it is well worth considering when solving large stochastic models.

The availability of fast and reliable methods for solving stochastic models paves the way to researchers who wish to explore frameworks that are much more complex, and much more interesting, than the one analyzed in these introductory notes.

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12. Appendix: the numerical routines

The Matlab routines used in these notes can be downloaded **here.** [50] Eleven routines are packed in a zip file.

In this Appendix, we detail the task performed by each routine. You can find some more comments and suggestions within each script file.

- valfun_iter0.m: this is the script (i.e. a list of command) that produces Table 1
- valfun_iter1.m: this script is essentially identical to valfun_iter0.m, but for the fact that we consider a grid of 1600 points; the state variable is ranging from 0.7 to 1.1. This routine yields Figure 5.
- valfun_iter2.m: this script performs the value function iteration procedure described in Sub-section 10.1. There are 800 gridpoints for capital (ranging again from 0.7 to 1.1) and five equally-spaced and equiprobable productivity levels (going from 3.367697 to 3.505155). This routine yields Figure 11. Table 2 is obtained by changing the number of gridpoints for the capital stock and the number of productivity levels.
- colloc_0.m: this script approximates $\sin(x)$ in the interval $[0, 2\pi]$ using first a third, and then a tenth degree polynomial. Figure 6 and 7 are the output for this script. Notice that this script calls for the function "primitive(x,num)". This is a function written in a separated file (also called a "function m-file").
- primitive.m: this function file stores the function primitive(x,num), which can be used by other Matlab files. This file produces a row vector of length num, the elements of which take values x^0, x^1, x^2, ..., $x^{(num-1)}$. For example, if a script file calls primitive(2,5), the output of the file primitive.m will be $[1, 2, 4, 8, 16]$. Notice that if we multiply the output of primitive.m by a column vector of parameters (of appropriate dimension), we obtain the value of the polynomial of degree num, characterized by the parameters' values provided by the column vector, and computed at x. Typically, in our exercises, x is a collocation point and hence $x \in [0, 2\pi]$.
- colloc_1.m: this script approximates the differential equation (6.4) for $t \in [0, 4]$ using first a second, and then an eight degree polynomial, and it produces Figures 8 and 9. This script uses the functions "primitive(x,num)" and "deriv_1(x,num)".

[50] I.e. at: http://www3.unicatt.it/pls/unicatt/consultazione.mostra_pagina?id_pagina=7223

- deriv_1.m: this file stores the function deriv_1(x,num). This file produces a row vector of length num, the elements of which take values 0, 1, $2x$, ..., $(num-1)\ x^{(num-2)}$. For example, if a script file calls deriv_1(2,5), the output of the file deriv_1.m will be $[0, 1, 4, 16, 32]$. Notice that if we multiply the output of deriv_1.m by a column vector of parameters (of appropriate dimension), we obtain the first derivative of the polynomial of degree num, characterized by the parameters' values provided by the column vector, and evaluated at x.
- colloc_2.m: this file solves system (6.8-6.9). This routine exploits the Matlab built-in nonlinear equation solver. This is done through the command fsolve(.). System (6.8-6.9) is stored in the files system0.m, and k1.m; Figure 10 represents the output for this script.
- system0.m. In this function file, we store eleven equations, eleven being the number of the d_is coefficients to be determined. Ten equation are of the type $\sum_{i=0}^{11} d_i k_1^i = \sum_{i=0}^{11} d_i k_0^i \left(k_1^{-0.7} + 0.8245\right)$, while the eleventh one is the steady state relation $\sum_{i=0}^{11} d_i \hat{k}^i = 3.43643\hat{k}^{0.3} - 0.85\hat{k}$. Because the k_1^is are endogenous (recall the second equation in (6.8)), each of them is determined as a function of the corresponding k_0^i and $c_0^i \left(= \sum_{i=0}^{11} d_i k_0^i\right)$, by the k1.m function file.
- k1.m. This function file stores the second equation composing (6.8), hence it determines the k_1^i as function of the parameters (A, α, β, and δ), of the k_0^i (identified in this routine by the i-th element of the vector var_kap) and by the vector composed of the num parameters d_is (this file reads the d_is as vector x).
- PEA_rmsy.m: this exemplifies the Parameterized Expectations Approach described in Sub-section 10.2; it produces Figure 12.

TABLE 1

iter.#	$V(k_0{=}0.98)$	$V(k_0{=}0.99)$	$V(k_0{=}1.00)$	$V(k_0{=}1.01)$	$V(k_0{=}1.02)$
0	0	0	0	0	0
1	0.890218	0.894487	0.898707	0.902881	0.907009
2	1.753757	1.758043	1.762281	1.766486	1.770648
3	2.591406	2.595692	2.599944	2.604152	2.608314
4	3.403926	3.408223	3.412478	3.416686	3.420850
5	4.192081	4.196381	4.200636	4.204844	4.209008
6	4.956594	4.960894	4.965149	4.969357	4.973521
7	5.698172	5.702472	5.706727	5.710935	5.715099
8	6.417502	6.421802	6.426058	6.430265	6.434430
9	7.115253	7.119553	7.123808	7.128016	7.132180
10	7.792071	7.796371	7.800626	7.804834	7.808998
20	13.538224	13.542524	13.546779	13.550987	13.555151
50	23.204550	23.208850	23.213105	23.217313	23.221477
100	28.264686	28.268986	28.273241	28.277449	28.281613
200	29.608749	29.613049	29.617304	29.621512	29.625676
300	29.672662	29.676962	29.681218	29.685425	29.689590
375	29.675528	29.679828	29.684084	29.688291	29.692456
376	29.675538	29.679838	29.684093	29.688301	29.692465
true	29.675860	29.680156	29.684409	29.688619	29.692788

TABLE 2

iter. #	$\mathrm{E}[V(0.98,A)]$	$\mathrm{E}[V(0.99,A)]$	$\mathrm{E}[V(1.00,A)]$	$\mathrm{E}[V(1.01,A)]$	$\mathrm{E}[V(1.02,A)]$
0	0	0	0	0	0
1	0.889825	0.894094	0.898316	0.902490	0.906619
2	1.753071	1.757366	1.761613	1.765815	1.769976
3	2.590458	2.594755	2.599010	2.603218	2.607381
4	3.402731	3.407029	3.411284	3.415492	3.419656
5	4.190637	4.194935	4.199190	4.203398	4.207563
10	7.789475	7.793774	7.798029	7.802236	7.806402
50	23.197025	23.201323	23.205579	23.209786	23.213952
100	28.255542	28.259841	28.264096	28.268304	28.272469
200	29.599175	29.603474	29.607729	29.611937	29.616102
300	29.663068	29.667367	29.671622	29.675830	29.679995
375	29.665933	29.670232	29.674487	29.678695	29.682860
376	29.665943	29.670242	29.674497	29.678705	29.682870
true	29.666455	29.670751	29.675004	29.679214	29.683383

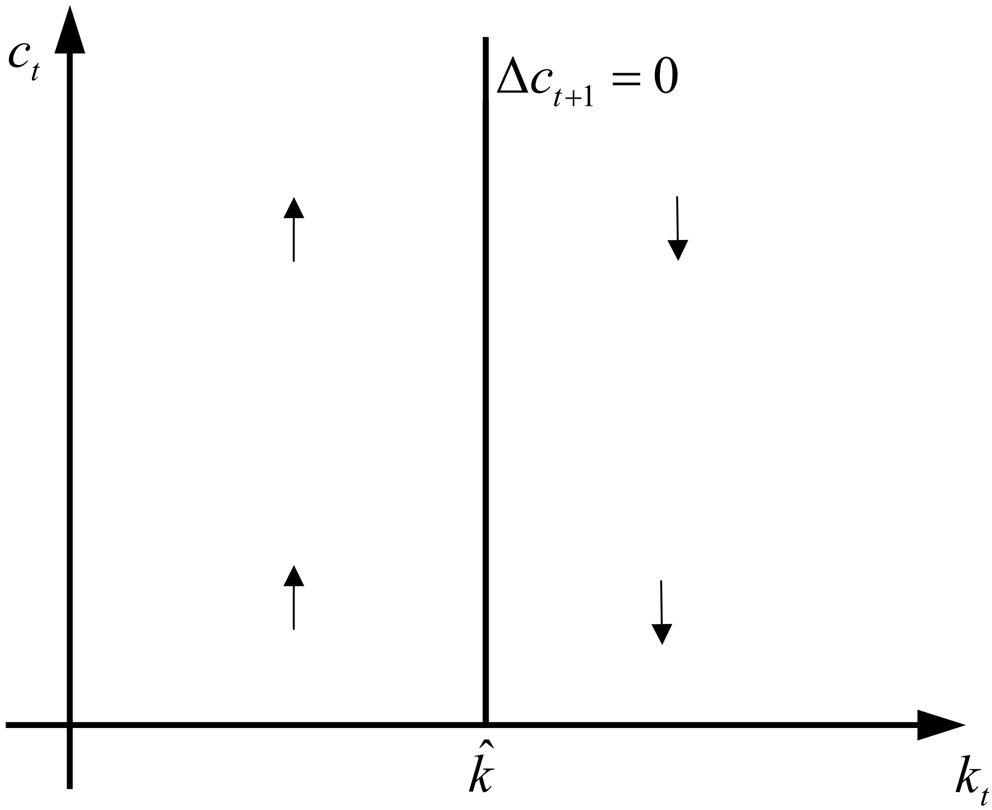


Figure 1

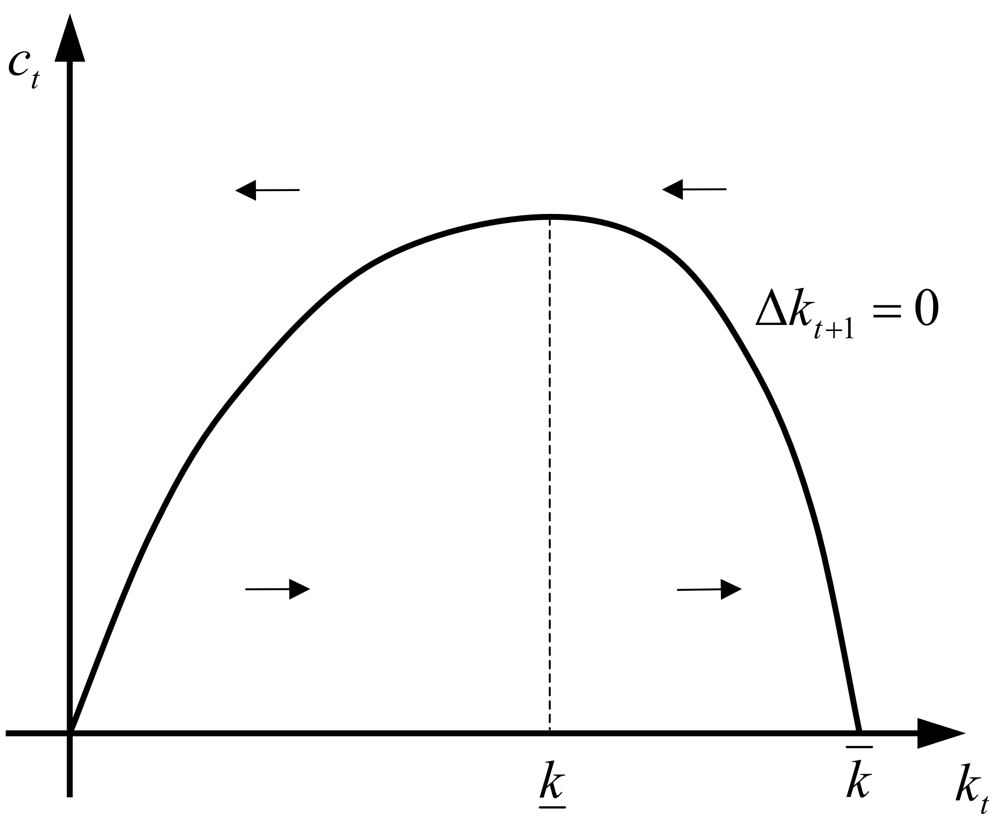


Figure 2

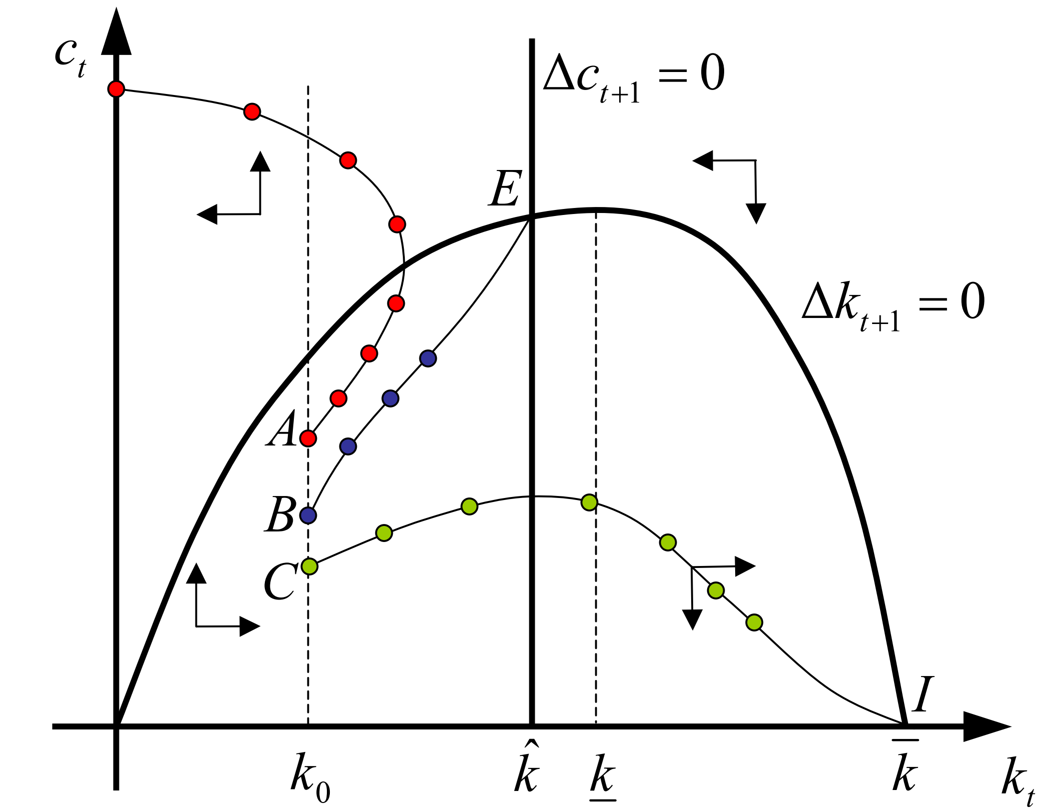


Figure 3

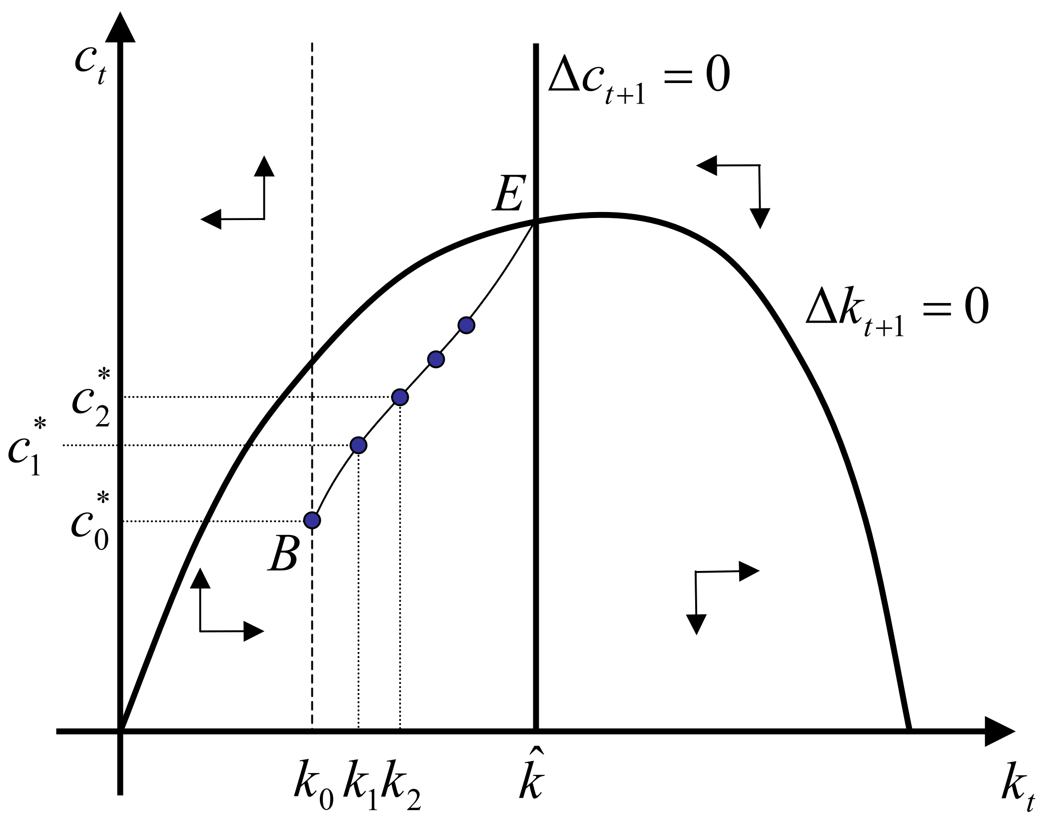


Figure 4

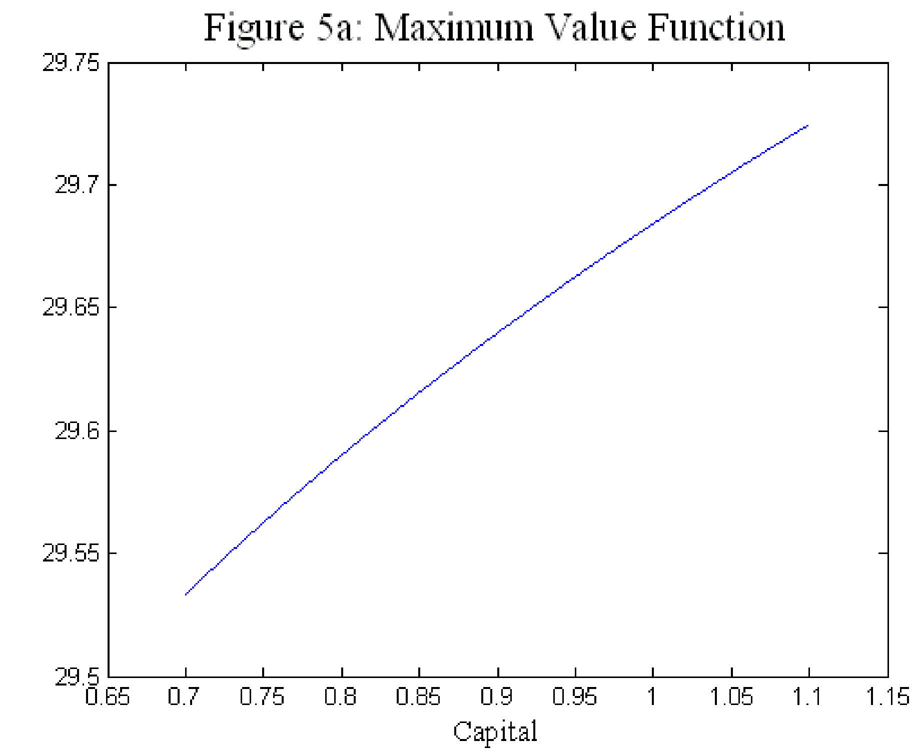


Figure 5a: Maximum Value Function
29.75
29.7
29.65
29.6
29.55
29.5
0.65
0.7
0.75
0.8
0.85
0.9
0.95
1
1.05
1.1
1.15
Capital

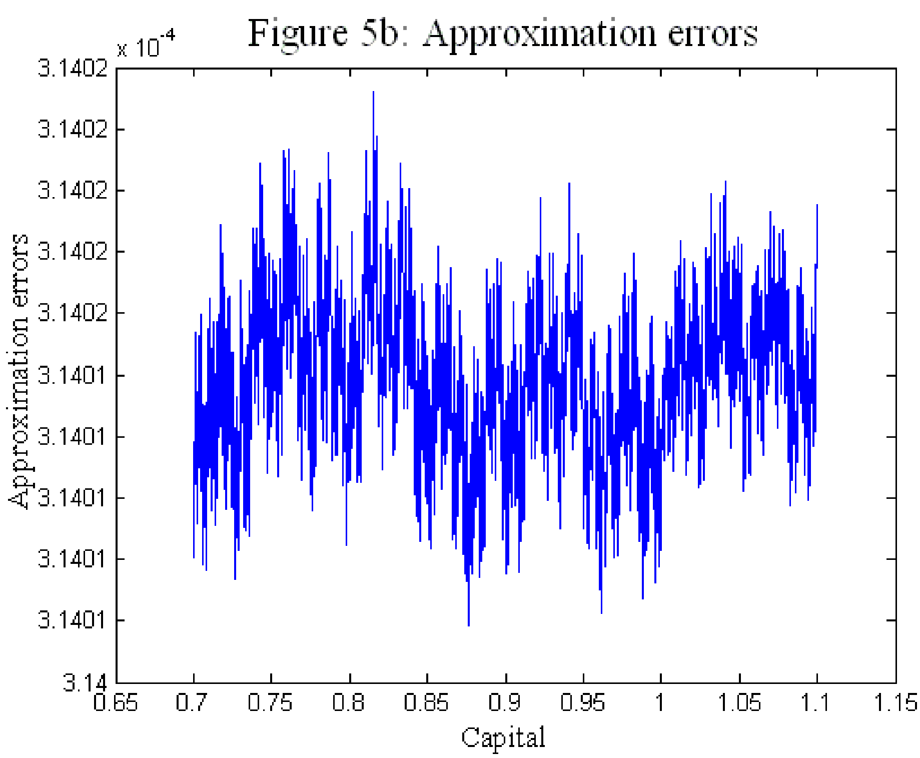


Figure 5b: Approximation errors
x 10^-4
3.1402
3.1402
3.1402
3.1402
3.1402
3.1401
3.1401
3.1401
3.1401
3.1401
3.14
Approximation errors
0.65
0.7
0.75
0.8
0.85
0.9
0.95
1
1.05
1.1
1.15
Capital

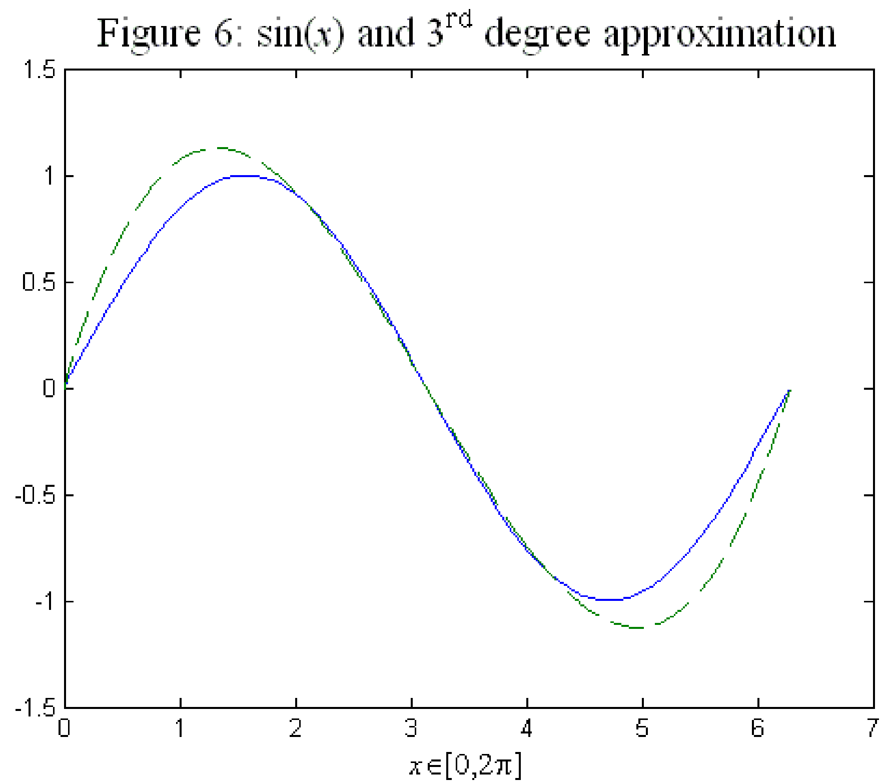
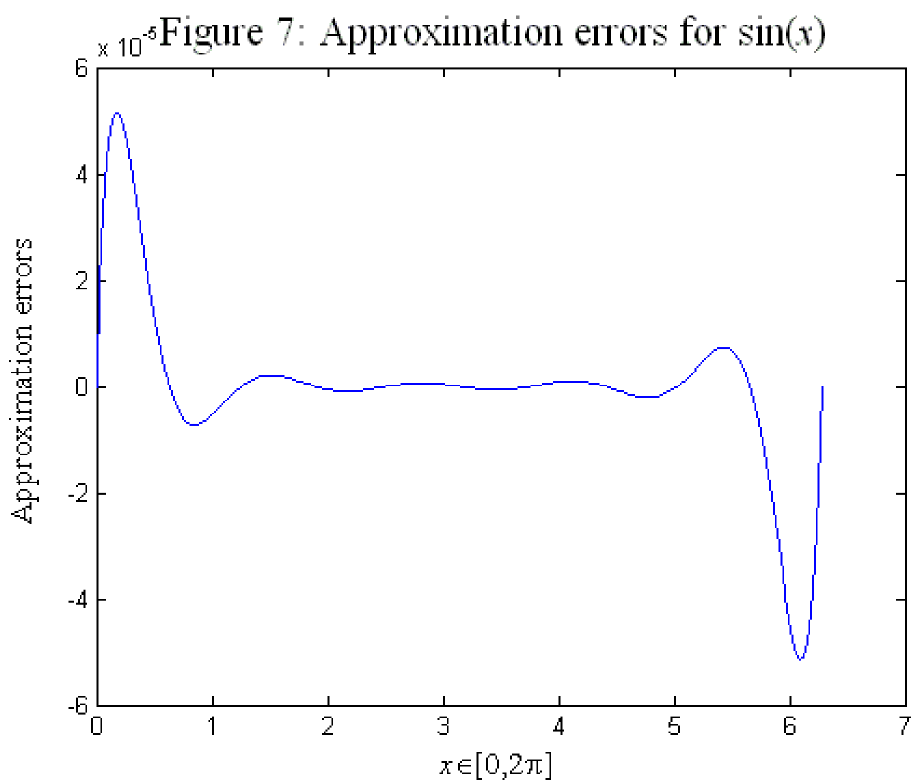


Figure 6: sin(x) and 3rd degree approximation
1.5
1
0.5
0
-0.5
-1
-1.5
0
1
2
3
4
5
6
7
x∈[0,2π]


x 10-5 Figure 7: Approximation errors for sin(x)
6
4
2
0
-2
-4
-6
0
1
2
3
4
5
6
7
Approximation errors
x∈[0,2π]

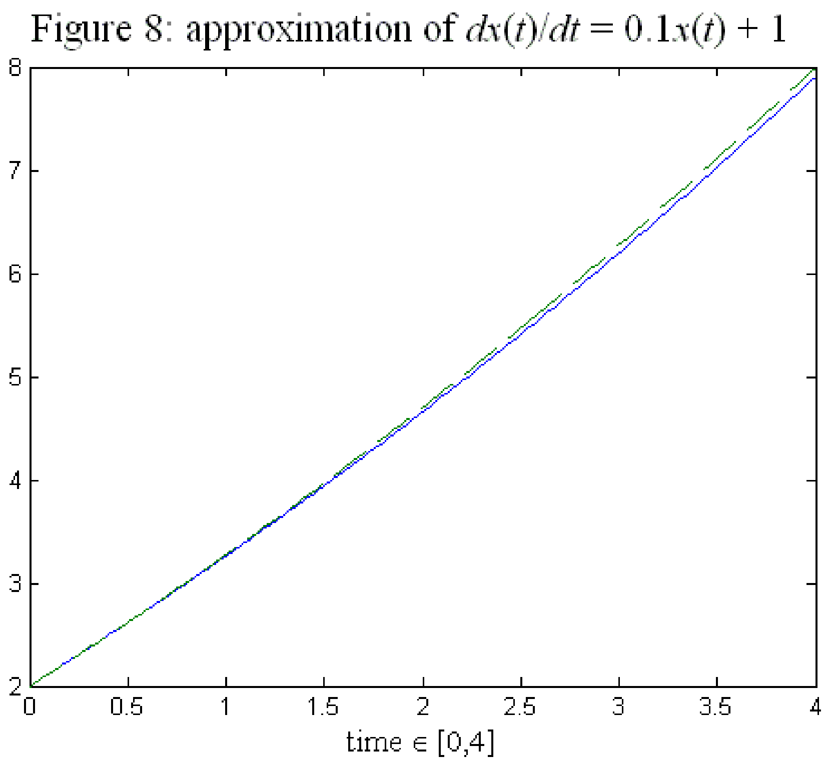


Figure 8: approximation of $dx(t)/dt = 0.1x(t) + 1$
8
7
6
5
4
3
2
0
0.5
1
1.5
2
2.5
3
3.5
4
time ∈ [0,4]

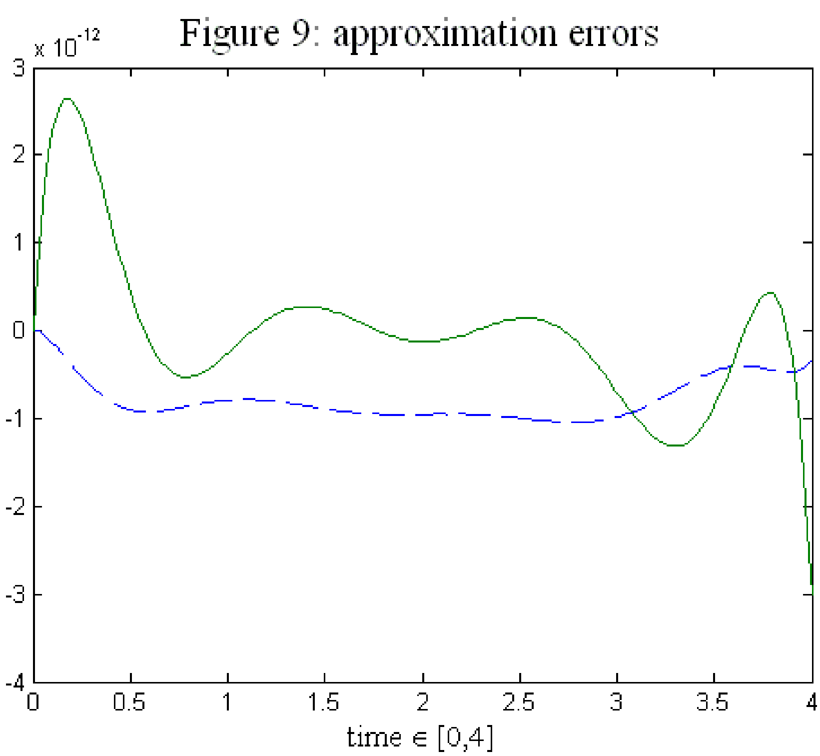


Figure 9: approximation errors
x 10^{-12}
3
2
1
0
-1
-2
-3
-4
0
0.5
1
1.5
2
2.5
3
3.5
4
time ∈ [0,4]

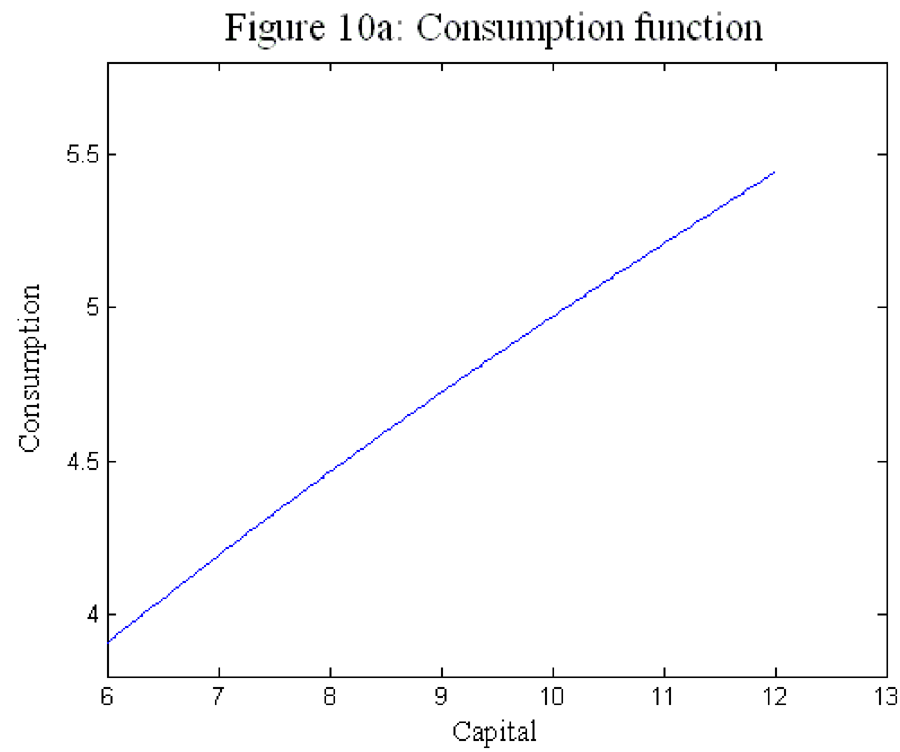


Figure 10a: Consumption function
Consumption
Capital
4
4.5
5
5.5
6
7
8
9
10
11
12
13

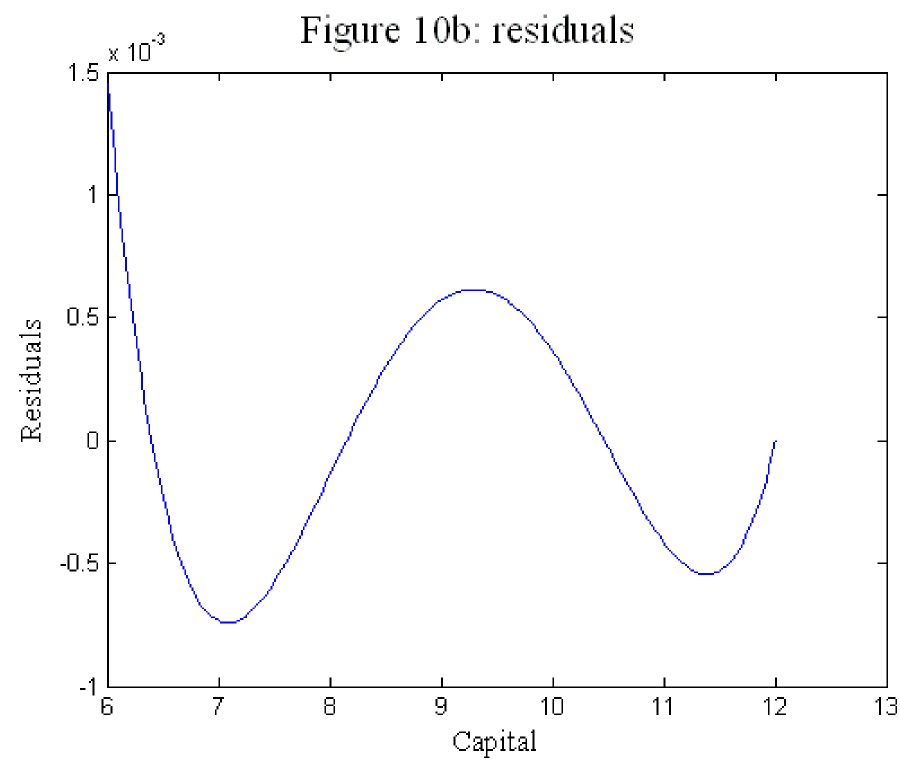


Figure 10b: residuals
x 10^-3
Residuals
Capital
-1
-0.5
0
0.5
1
1.5
6
7
8
9
10
11
12
13

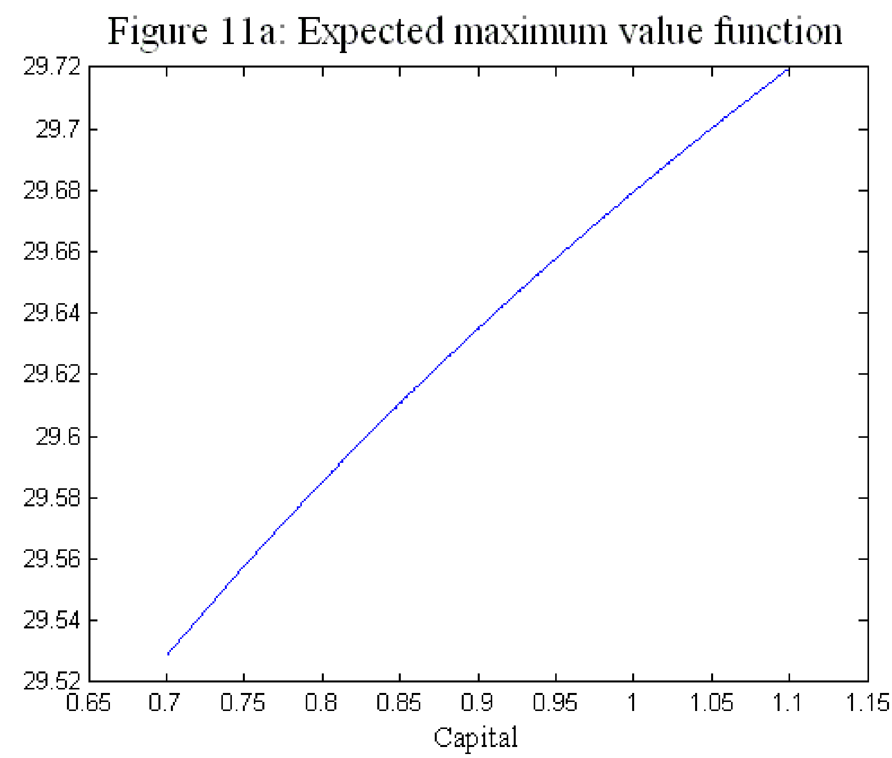
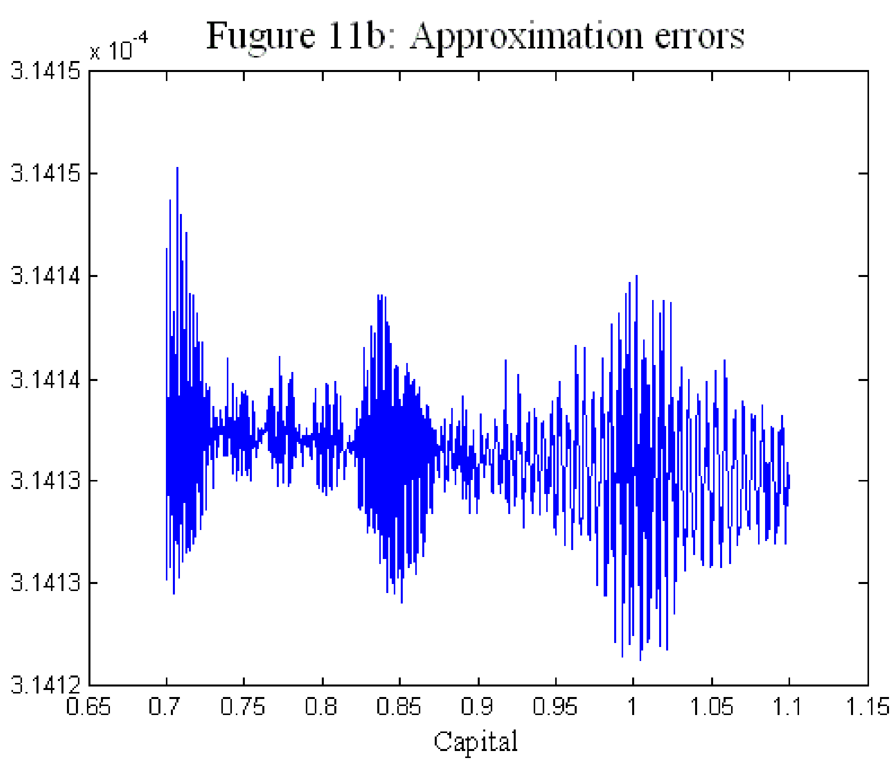


Figure 11a: Expected maximum value function
29.72
29.7
29.68
29.66
29.64
29.62
29.6
29.58
29.56
29.54
29.52
0.65
0.7
0.75
0.8
0.85
0.9
0.95
1
1.05
1.1
1.15
Capital


Fugure 11b: Approximation errors
x 10^-4
3.1415
3.1415
3.1414
3.1414
3.1413
3.1413
3.1412
0.65
0.7
0.75
0.8
0.85
0.9
0.95
1
1.05
1.1
1.15
Capital

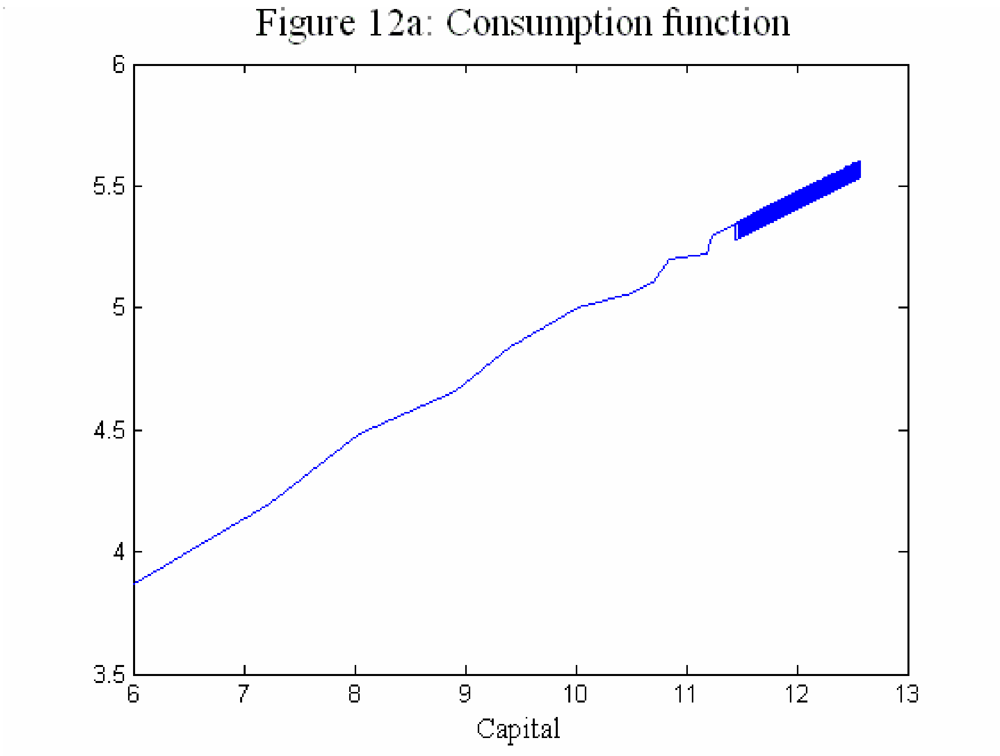


Figure 12a: Consumption function
6
5.5
5
4.5
4
3.5
6
7
8
9
10
11
12
13
Capital

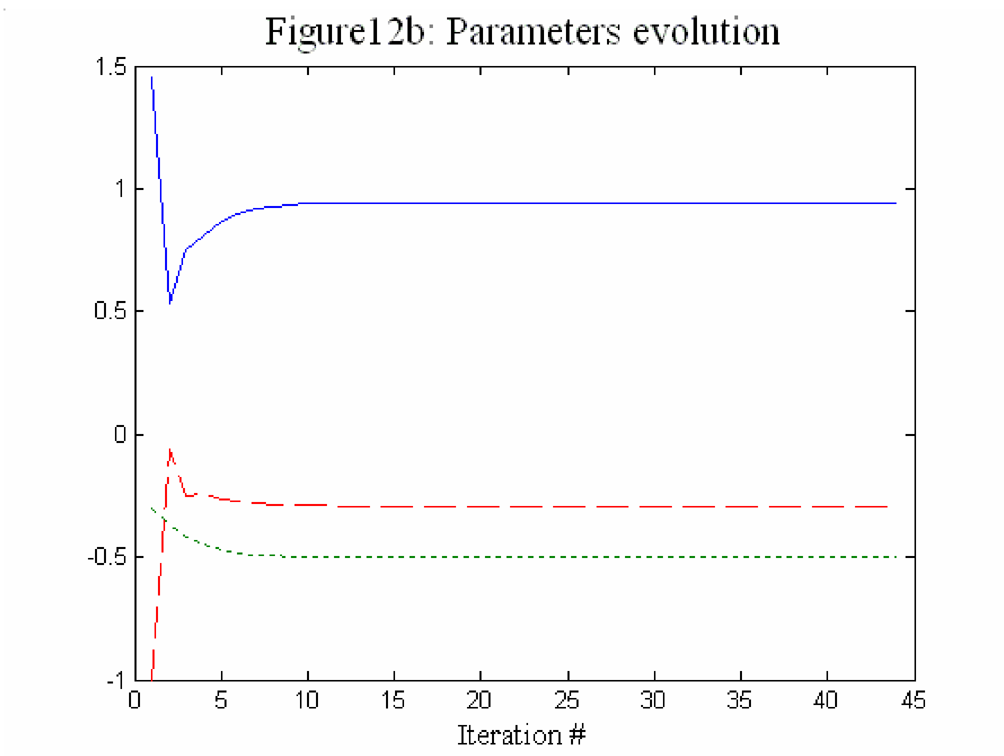


Figure12b: Parameters evolution
1.5
1
0.5
0
-0.5
-1
0
5
10
15
20
25
30
35
40
45
Iteration #